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ON THE GEOMETRY OF LUXURY

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Parole chiave: Duality, homotheticity, Cobb-Douglas function, luxury, expansion path, elasticity of substitution, scale effect, substitution effect, income effect, Shephard's distance.

On the geometry of luxury

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Abstract *A 2-parameter class of ordinal utility functions over a pair of goods is discussed with respect to general traits of preferences for luxury. The class contains Cobb-Douglas functions as no-luxury limit; its analytical tractability is probed by simple closed form solutions for Marshallian demand functions, expansion paths, Engel curves, income elasticity of demand, saturation levels, elasticity of substitution. Following Mantovi* (2013)*, scale and substitution effects can be represented in terms of flows on bundle space; departure from homotheticity can thereby be represented by an index of luxury which measures the noncommutativity of such effects. On conceptual grounds, our index is intimately connected with Shephard's distance. Decompositions of productive efficiency as tailored by Bogetoft et al.* (2006) *represent a natural setting for the application of our approach.*

This draft, June 28

1 Introduction

Cobb and Douglas (1928) set forth a landmark theory of aggregate manufacturing production in terms of highly tractable production functions. Over the decades, the Cobb-Douglas (CD) functional form has proved relevant as well in the representation of preferences, so as to become "perhaps the most ubiquitous function in all of economics" (Chambers, 1988). Still, being homothetic,[1] CD functions cannot represent general traits of preferences like the luxury/necessity dichotomy, namely, the case for the relative consumption of a luxury good to increase with income as compared to a necessary good. It is the twofold aim of the present contribution to (*i*) address a smooth generalization of CD utility functions as a general representations of the luxury/necessity dichotomy, and to (*ii*) define an index of luxury, building on the differential geometric framework set forth by Mantovi (2013). The author argues about the relevance of primal parametrizations of income effects[2] (INE) and substitution effects (SUE), establishes that such effects (once properly parametrized) do commute for homothetic models, and that the Lie bracket of such flows provides a local measure of noncommutativity. It turns out that an insightful index of luxury results as a measure of departure from the homothetic commutative benchmark. The index vanishes for homothetic consumers, and its interpretation is particularly transparent with respect to our class of preferences.

Recall, luxury/necessity pairs of goods are such that expansion paths bend towards the luxury axis (for instance, Varian, 1992), so that income elasticity of demand for the luxury good exceeds 1. Typical examples of luxury goods are jewelry and foreign travel, whereas typical examples of necessary goods are foodstuff (for instance, Frank, 2010). The luxury/necessity dichotomy represents a general trait of preferences, to the extent that homothetic preferences represent the benchmark case in which the relative composition of optimal consumption bundles is invariant with respect to INE, and income elasticity of demand is unitary for all goods.

True, one can envision a number of patterns for expansion paths to foliate bundle space. At one extreme, a luxury/necessity pattern may represent a transient phase,

[1] The introduction of homotheticity in economic theory is attributed to Shephard (1953), as pointed out for instance by Chambers and Mitchell (2001).

[2] By *expansion* effects the author denotes both income and output effects.

followed by a phase in which, for large enough income, the luxury good becomes superior with respect to the necessary-turned-inferior[3] good. At another extreme, the luxury to necessity ratio may stabilize to a finite value in the large income limit, or expansion paths may display oblique asymptotes, i.e. luxury may 'decay' with income. In between such opposite extremes, our model represents the general case in which the necessary good does not become inferior, and a saturation (satiation of consumption) of the necessary good occurs, represented by upper bounds for Engel curves. As such, our model seems to embody general traits of the luxury/necessity dichotomy, and combines the analytical tractability of CD models with the flexibility guaranteed by the parameter ε, which endows the good *y* with increasing luxury. Basically, we tailor a map of luxury with coordinate ε, which contains CD models as $\varepsilon \to 0$ limit.

We shall address the luxury/necessity dichotomy in terms of a differential geometric approach meant to tailor a sharp perspective on duality.[4] According to Gorman (1976), "Duality is about the choice of the independent variables in terms of which one defines a theory." Definitely, we shall introduce flows on the space of primal (endogenous) variables, meant to characterize the solution of the consumption problem: given the equivalence of INE and SCE in homothetic models, we shall take the departure from such equivalence as a measure of luxury. Such an approach proves both *effective*, as witnessed by the results set forth by Mantovi (2013), and *insightful*, as witnessed for instance by the transparent algorithm for the computation of elasticity of substitution (formula 26 below).

Basically, our approach to scale effects (SCE – homotheties) on bundle space aligns with the philosophy embodied by Shephard's distance $\mathcal{D}$. Recall, $\mathcal{D}$ provides a representation of preferences[5] (and, correspondingly, of production possibilities) in

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[3] Bertoletti and Rampa (2013) set forth a new class of production functions exhibiting inferior inputs, which may perhaps represent preferences for superior/inferior goods.

[4] For thorough accounts of duality theory we refer to Chambers (1988) and Cornes (1992), according to whom a geometric approach helps convey "an intuitive understanding of duality theory." Along similar lines, in the author's vision, the following differential geometric approach conveys sharp insights about the properties of partial equilibrium which cannot be grasped in terms of comparative statics.

[5] As pointed out by Cornes (1992), purists like to define preferences in terms of $\mathcal{D}$, so as to purge their language of the somewhat unpleasant "cardinal overtones" which reverberate through utility functions.

terms of SCE: in essence, the distance between a bundle and an indifference curve is uniquely determined by the SCE which connects the bundle to the curve (the key to the definition of technical efficiency set forth by Farrell, 1957). Along similar lines, we shall employ SCE on bundle space as probes of luxury as departure from homotheticity. In fact, we shall find out that our index is intimately connected with $\mathcal{D}$ via a "deviation factor" from scale symmetry. Decompositions of overall productive efficiency as tailored by Bogetoft et al. (2006) represent a natural target for our approach, on account of the result set forth by Mantovi (2013) about the equivalence of standard and reversed Farrell decompositions for homothetic models.

The plan of the rest of the paper is as follows. In section 2 we define a class of preferences over a luxury and a necessary good, solve the budget-constrained optimal consumption problems, and represent expansion paths and Engel curves for a sample consumer. In section 3 we sketch the rationale for a number of indicators of departure from CD homotheticity. In section 4 we introduce our differential geometric approach, which we apply in section 5 to the computation of elasticity of substitution. In section 6 we define our index. We address in section 7 the link with Shephard's distance. A final section is meant to pin down the relevance of our approach, as well as potential lines of progress. Admittedly, the aim of the paper is eminently methodological; in such respects, a number of cross-references to results in production analysis will enable us to capitalize on the 'isomorphism' between producers' and consumer's problems. We are interested in smooth convex preferences over a pair of goods such that expansion paths foliate smoothly bundle space, as a generalization of CD problems. Write $\mathcal{B} = (0,\infty)\times(0,\infty)$ for the manifold of consumption bundles with nonvanishing components, endowed with the natural differentiable structure. Assume all income is spent.

2 A class of preferences for luxury

Being the analytical tractability of CD models long established, it is tempting to address the luxury/necessity dichotomy in terms of a smooth generalization of CD preferences.

Thus, consider the 2-parameter class of preferences represented by the ordinal utility functions

$$U_{a,\varepsilon}(x,y) = x^a (ye^{\varepsilon y})^{1-a} \quad . \tag{1}$$

Such functions belong to the class of *transcendental* functions,[6] and then to the class of *generalized power* production functions defined by de Janvry (1972). The class (1) is parametrized by $a \in (0,1)$, generalizing the CD parameter, and by $\varepsilon \in [0,\infty)$, which 'injects' increasing luxury into good *y*, and drives a departure from the CD limit $\varepsilon \to 0$. The rationale for the choice of the functional form (1) is to factor out a term generating luxury (we may call it 'luxury effect'), so as to write utility functions as products of a CD function and an increasing function $F(y) = \exp(\varepsilon(1\text{–}a)y)$. *F* is chosen so that F' is proportional to *F*, in order to enhance the analytical tractability of the model. The budget-constrained optimal consumption problems defined by (1) are strictly convex, so that, for any set of exogenously given income and prices, an optimal consumption bundle is uniquely defined. We shall find out that such models lead to Marshallian demand for *y* not depending on the price of *x* (vanishing cross-price response).

Indifference curves for functions (1) can be represented as

$$x(y;u) = u^{\frac{1}{a}} (ye^{\varepsilon y})^{\frac{a-1}{a}} \quad . \tag{2}$$

Figure 1 represents sample indifference curves for $a = 0.75$, $\varepsilon = 0$ (CD) and $\varepsilon = 0.4$, corresponding to the utility levels $2e^{0.2} = U_{0.75,\,0.4}(2,2)$ and $4e^{0.4} = U_{0.75,\,0.4}(4,4)$. The luxury effect decreases for decreasing *y*, so that indifference curves for the same level *u* and different ε are seen to converge in the large *x* (small *y*) limit. The qualitative picture

[6] Such functions, in the case of two variables, are of the form $A x^a y^b \exp(cx+dy)$ for positive constants *A*, *a*, *b*, *c*, *d*; see for instance Chambers (1988).

emerging from Figure 1 provides a transparent representation of the luxury effect on *y*: MRS vary along rays so that, the larger the bundle, the less luxury is needed to substitute a definite amount of necessity; we thereby expect expansion paths to bend towards the luxury axis. It is not that straightforward to argue about cross-price responses: CD models display vanishing cross-price responses for all goods, an occurrence which we expect not to hold in our model. Let us address the analytics of the problem, in order to fix such properties.

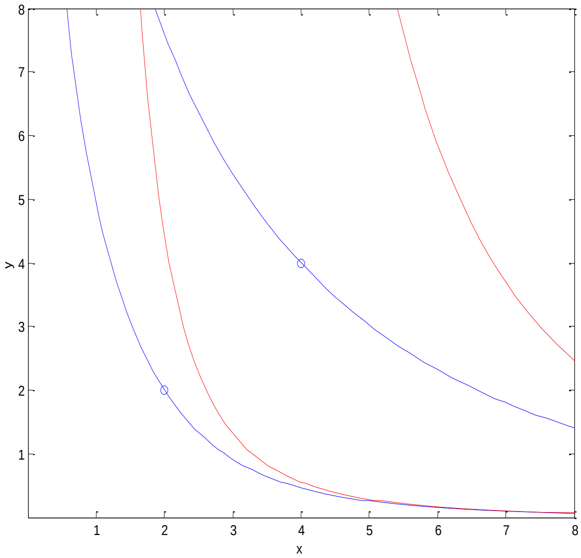

Figure 1 Indifference curves for utility levels $2e^{0.2}$ and $4e^{0.4}$ for a = ¾ , ε = 0 (Cobb-Douglas, steeper red curves) and ε = 0.4 (blue curves). The bundles (2,2) and (4,4) are represented.

Start with the condition for the differential (gradient) of *U* to be proportional to the price vector, i.e.

$$\begin{aligned} \frac{\partial U_{a,\varepsilon}}{\partial x} &= a e^{\varepsilon(1-a)y}\left(\frac{y}{x}\right)^{1-a} = a\frac{U_{a,\varepsilon}}{x} = \lambda p_x \\ \frac{\partial U_{a,\varepsilon}}{\partial y} &= (1-a)e^{\varepsilon(1-a)y}\left(\left(\frac{x}{y}\right)^{a} + \varepsilon\, x^a y^{1-a}\right) = (1-a)\frac{U_{a,\varepsilon}}{y}(1+\varepsilon y) = \lambda p_y \end{aligned} \tag{3}$$

from which we derive the “price ratio = MRS” condition

$$MRS = \frac{\frac{\partial U}{\partial y}}{\frac{\partial U}{\partial x}} = \frac{p_y}{p_x} = \frac{1-a}{a}\frac{x}{y}(1+\varepsilon y) \quad \Rightarrow \quad xp_x = \frac{a}{1-a}\frac{yp_y}{1+\varepsilon y} \quad . \tag{4}$$

Condition (4) is a smooth generalization of the corresponding CD condition, and provides a cartesian equation for expansion paths. Such a condition is independent of the utility representation; for instance, it can be obtained by the logarithmic representation $\widetilde{U}_{a,\varepsilon}(x,y) = \ln U_{a,\varepsilon}(x,y) = a\ln x + (1-a)\ln y + (1-a)\varepsilon y$ of utility. Then, the following results have ‘intrinsic’ significance. The budget constraint can be written

$$I = xp_x + yp_y = yp_y\left(1 + \frac{a}{1-a}\frac{1}{1+\varepsilon y}\right) \quad , \tag{5}$$

from which we derive closed form solutions for Marshallian demand functions. Marshallian demand for *y* can be represented as solution to the quadratic equation

$$\varepsilon\, y^2 + \left(\frac{1}{1-a} - \varepsilon\frac{I}{p_y}\right)y - \frac{I}{p_y} = 0 \quad . \tag{6}$$

Notice, *I* and p_y enter such an expression via their ratio, so as to comply with homogeneity of degree 0 in prices and income. Equation (6) admits a positive solution (the relevant one) and a negative solution, which read

$$y_{1,2} = \frac{1}{2\varepsilon}\left(-\frac{1}{1-a} + \varepsilon\frac{I}{p_y} \pm \sqrt{\left(\frac{1}{1-a} - \varepsilon\frac{I}{p_y}\right)^2 + 4\varepsilon\frac{I}{p_y}}\right) \quad , \tag{7}$$

with the positive solution $y^*(I,p_y)$ approaching the corresponding CD solution for $\varepsilon \to 0$.[7] Noticeably, Marshallian demand for y does not depend on p_x, thereby generalizing the corresponding CD property of vanishing cross-price response. Then, Marshallian demand for x can be written

$$x^*(I, p_x, p_y) = \frac{I}{p_x} - \frac{p_y}{p_x} y^*(I, p_y) \tag{8}$$

with the proper CD limit for $\varepsilon \to 0$, and with non vanishing cross-price response for positive ε. Out of (7) and (8) we derive the closed form solution for indirect utility as function of prices and income. It is not difficult to convince oneself that both laws of over- and undercompensated demand (Cornes, 1992) hold.

We can combine the transparency of the closed form solutions (7), (8) with the qualitative picture represented in Figure 1 in order to establish basic properties of our model, like Engel curves and expansion paths. In first instance, the functions $U_{a,\varepsilon}$ approach the CD limit $U_{a,0}$ for small y, so that for small I we expect expansion paths to approach the CD straight expansion paths represented in cartesian form by

$$x = \tfrac{a}{1-a} \frac{p_y}{p_x} y \quad . \tag{9}$$

We can check the properness of such an insight by exploring the representative case a = 0.75, ε = 0.4. Consider unit prices $p_x = p_y = 1$, and establish optimal consumption as a function of $I = x + y$. Solve the budget constraint for $x = I - y$ and then the problem $y^* = \arg\max\ (I-y)^{0.75} y^{0.25} e^{0.1y}$ in terms of the larger solution of the second order algebraic equation

[7] For instance, one may Taylor expand in ε the square root and identify the leading term of the positive root as the familiar CD Marshallian demand.

$$0 = y^2 + (10 - I)y - 2.5I \quad . \tag{10}$$

Thus, the analytical tractability of our models is probed by the following closed form solutions for Engel curves, namely,

$$y(I) = \frac{1}{2}\left(I - 10 + \sqrt{(I-10)^2 + 10I}\right) \tag{11}$$

for the luxury good, and

$$x(I) = \frac{1}{2}\left(I + 10 - \sqrt{(I-10)^2 + 10I}\right) \tag{12}$$

for the necessary good. Such functions are plotted in the upper part of Figure 2.

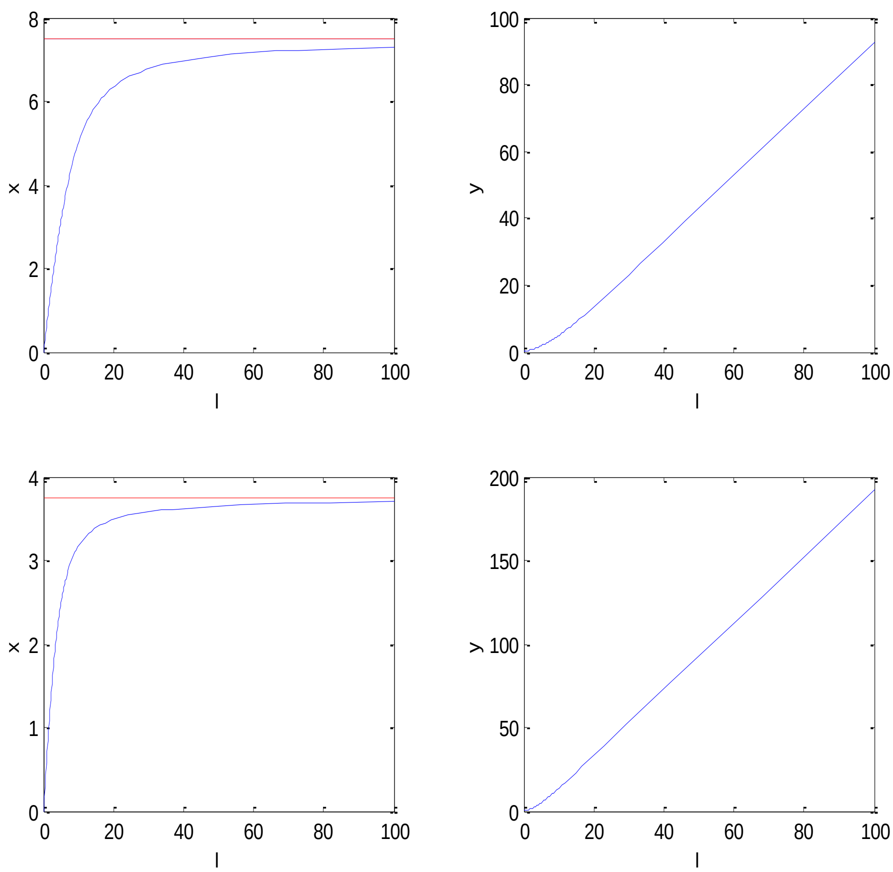

Figure 2 Engel curves for p_x = 1, p_y = 1 (upper) and p_y = 0.5 (lower), with α = 0.75 and ε = 0.4. The Engel curves for the necessary good display asymptotes according to (13). The Engel curves for the luxury good display asymptotic derivative $1/\,p_y$.

We can employ formulas (11) and (12) as parametric representation of the corresponding expansion path, which, on account of (4), admits the cartesian representation $x = \dfrac{3y}{1+0.4y}$. For small *I* we expect such expansion path to approach the corresponding CD path; in fact, for *I* = 0.1 we have $y(0.1) \approx 0.025188$, a tiny deviation from the corresponding CD solution $y(I) = 0.25I$. Then, for increasing income, our solutions (11), (12) define a transparent representation of INE, as plotted in Figure 3 (rightmost curve).

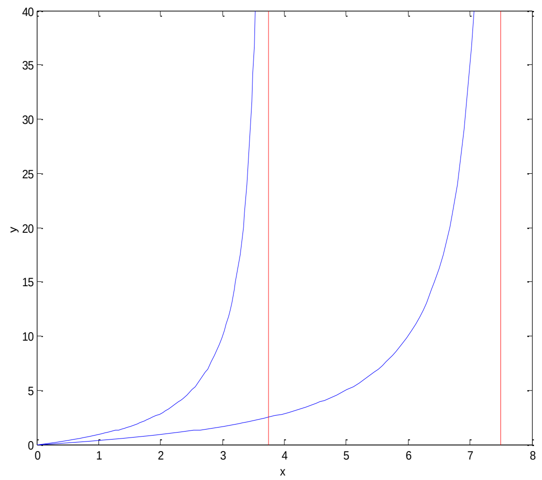

Figure 3 Expansion paths for $p_x = 1$, $p_y = 1$ (rightmost curve) and $p_y = 0.5$ (leftmost curve), with $a = 0.75$ and $\varepsilon = 0.4$. The vertical asymptotes correspond to the asymptotes of *x* Engel curves in Figure 2.

The same qualitative behavior occurs for all price ratios. For instance, for $2p_x = p_y = 1$, the formula for the *y* Engel curve is the same as (11), on account of vanishing cross-price response, whereas the formula for the *x* Engel curve reads $x(I) = I + 10 - \sqrt{(I-10)^2 + 10I}$. Correspondingly, for $p_x = 2p_y = 1$, it follows from the equation $0 = y^2 + (10-2I)y - 5I$ for *y* Marshallian demand that the formulas for Engel curves read $y(I) = \dfrac{1}{2}\left(2I - 10 + \sqrt{(2I-10)^2 + 20I}\right)$ and $x(I) = \dfrac{1}{2}\left(I + 5 - 0.5\sqrt{(2I-10)^2 + 20I}\right)$, whose plots are represented in the lower part of Figure 2, displaying a different saturation level. The corresponding expansion path is represented in Figure 3 (leftmost curve). In fact, a sharp characterization of our model is provided by the saturation levels for the necessary good. Take the large *y* limit of (4) and find out that such levels are given by

$$\bar{x} = \tfrac{a}{1-a} \frac{p_y}{p_x} \frac{1}{\varepsilon} \quad . \tag{13}$$

Notice the transparent characterization of such levels as function of a, ε and price ratio: the levels (13) diverge in the $\varepsilon \to 0$ limit in which the luxury effect vanishes, and vanish in the opposite limit in which luxury 'explodes.' Furthermore, such levels are proportional to the price ratio, so that, for instance, the luxury effect $\varepsilon \to 2\varepsilon$ can be exactly offset by the price effect $p_y \to 2\, p_y$.

Then, we are in position to employ Hotelling-Wold identities (for instance, Cornes, 1992) in order to write down the closed form solution for inverse demand functions, namely,

$$\frac{p_y}{I} = \frac{1-a}{y(1+(1-a)\varepsilon y)} \equiv \frac{1+\varepsilon y}{\chi_{a,\varepsilon}(y)} \frac{1-a}{y} \;, \quad \frac{p_x}{I} = \frac{a}{x(1+(1-a)\varepsilon y)} \equiv \frac{1}{\chi_{a,\varepsilon}(y)} \frac{a}{x} \tag{14}$$

so as to satisfy the identity $y\frac{p_y}{I} + x\frac{p_x}{I} = 1$. As expected, corresponding to the vanishing cross-price response of (7), inverse demand for the luxury good does not depend on the amount of consumption of the necessary good. Noticeably, inverse demand for the necessary good is obtained simply by dividing the CD inverse demand *a/x* by the *deviation factor* $\chi_{a,\varepsilon}(y) \equiv 1 + (1-a)\varepsilon y$, whose geometric characterization we shall address in section 4. It is not difficult to employ formulae (14) to establish the own price elasticities η_x , η_y of Marshallian demand functions; definitely, as a generalization of CD models, η_x is unitary. Figure 4 pictures functions (14) for varying 'intensity' of luxury.

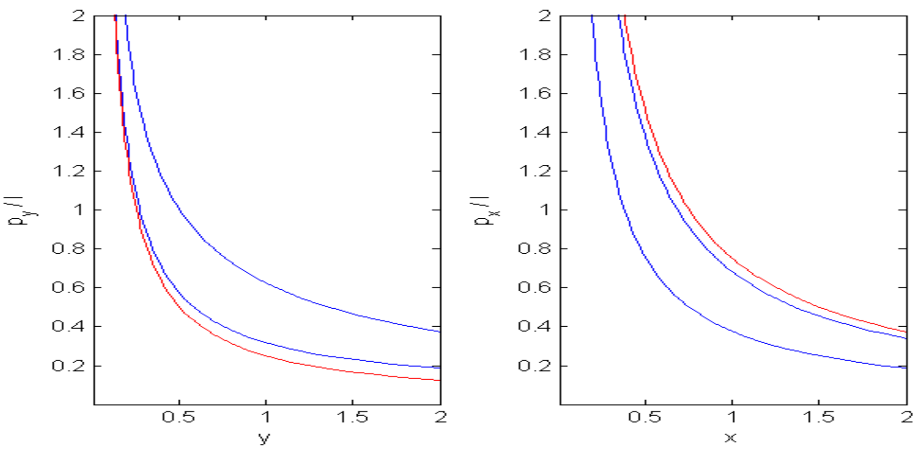

Figure 4 Inverse demand functions (14) for the luxury good (left) and for the necessary good (right, y = 1) for a = 0.75 and luxury 'intensity' ε = 0 (CD red curves), 0.4, 4 (the larger ε, the higher the curve for the luxury good, the opposite for the necessary good).

Indeed, we will find that the deviation factor represents the 'core' of our model (1) of preferences. For instance, consider the closed form solution for the marginal utility of money (income): it follows from (3) that $\lambda I = \lambda(xp_x + yp_y) = (1+(1-a)\varepsilon y)U_{a,\varepsilon}$, so that $\lambda_{a,\varepsilon} = \chi_{a,\varepsilon}(y)\dfrac{U_{a,\varepsilon}(x,y)}{I}$. Then one can employ the Marshallian demand functions in order to obtain the standard representation of marginal utility of money as function of prices and income.

To sum up, the following proposition crystallizes main features of our model (1) of preferences.

Proposition 1 *The class* (1) *of preferences displays Marshallian demand for the luxury good which does not depend on the price of the necessary good. Engel curves are upward sloping for both goods for all levels of income. Engel curves for the necessary good display asymptotes* $\bar{x} = \dfrac{a}{1-a}\dfrac{p_y}{p_x}\dfrac{1}{\varepsilon}$, *representing satiation of necessity for positive* ε. *Engel curves for the luxury good have asymptotic derivative* $1/p_y$ *since, asymptotically, "luxury takes it all" (income). The deviation factor* $\chi_{a,\varepsilon}(y) \equiv 1+(1-a)\varepsilon y$ *measures the divergence from the CD scale symmetry embodied by limit* $\varepsilon \to 0$.

3 Departure from CD homotheticity

We are in a position to capitalize on the analytical tractability of our model in order to argue about the significance of various measures of departure from CD homothetic consumption, as motivation for our approach. In first instance, once we address luxury in terms of bending of expansion paths, a natural indicator of luxury is the convexity of expansion paths *x*(*y*) as measured by $\frac{d^2x}{dy^2}$. It follows from (4) that

$$\frac{d^2x}{dy^2} = \frac{p_y}{p_x}\frac{a}{1-a}\frac{-2\varepsilon}{(1+\varepsilon y)^3} \tag{15}$$

which, as expected, is negative for positive ε, thereby representing bending towards the *y* axis. Such an expression (independent of the utility representation) displays the correct vanishing limit for $\varepsilon \to 0$. Furthermore, such an expression displays the correct vanishing limit for large *y*, in which expansion paths approach their asymptotes. Indeed, (15) is a valid indicator of the luxury effect, and partially complies with our primal approach. Still, prices remain as labels of expansion paths, and INE are only implicitly represented.

A significant measure of departure from homotheticity is the elasticity of substitution σ (independent of the utility representation), which we expect not to be constant. We shall confirm such an insight in terms of an explicit closed form solution (formula 26 below), with the correct CD limit for $\varepsilon \to 0$. Recall that CD models display unit elasticity of substitution,[8] on account of the proportionality (9) between MRS and *y*/*x*; our model is meant in fact to map luxury in terms of departure from such a limit. True, elasticity of substitution provides information about the slope of indifference curves as function of *y*/*x*, and only implicitly about the bending of expansion paths.

[8] Arrow et al. (1961) motivate the introduction of a new class of production functions with the need to escape the straitjacket of zero or unit elasticity of substitution.

Definitely, income elasticity of demand exceeding 1 is the standard indicator of a good being a luxury. As a probe of the analytical tractability of our model, notice that income elasticity of demand for luxury (11) results in

$$\frac{I}{y^*}\frac{dy^*}{dI} = I\,\frac{1+\dfrac{I-10+5}{\sqrt{(I-10)^2+10I}}}{I-10+\sqrt{(I-10)^2+10I}} \tag{16}$$

which exceeds 1 for positive *I*.

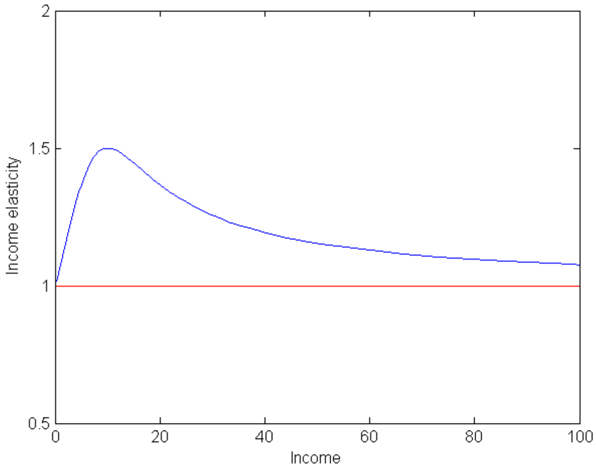

Figure 5 Income elasticity of demand for the luxury good for unit prices, a = 0.75, ε = 0.4.

As represented in Figure 5, the function (16) displays a peak at I = 10 p_y , and then approaches asymptotically the value 1 from above. Evidently, for different expansion paths (different price ratio), the qualitative behavior is the same, with the peak increasing its height for increasing p_y/p_x and ε. Thus, income elasticity displays an increasing phase followed by a decreasing one, and in principle it is not clear which of its values should be taken as measure of luxury; true, the height of the peak measures the amount of bending of expansion paths for same ε and varying price ratio.

An interesting approach to 'measure' luxury is set forth by Moffatt and Moffatt (2011). Such Authors address reflexion properties of direct and indirect utility functions, and derive conditions for luxury/necessity patterns in terms of functions on bundle space (combination of derivatives of the direct utility function), thereby resembling, to some extent, our primal perspective on duality. In fact, the interpretation of such functions is tightly connected to the reflexion properties which drive their approach. On the other hand, our approach is meant to focus luxury as departure from the scale symmetry of homothetic models, and then from the commutativity of INE/SCE and SUE (Mantovi, 2013).

4 The differential geometry of duality

According to Chambers (1988), the essence of duality lies in the reconstruction of the original technology from the cost function; such a perspective is meant to focus the properties of cost functions as the essence of production analysis. On the other hand, our focus on consumption and luxury leads us to conceive INE and SCE on $\mathcal{B}$ as the building blocks of our arguments. Then, on conceptual grounds, our approach focuses the divergence of INE and SCE (Figure 6) as the essence of duality; on technical grounds, we shall employ vector fields on $\mathcal{B}$ as generators of (properly parametrized) SCE and SUE meant to probe such a divergence. Recall, the naturality of SCE-SUE decompositions of basic indices has long been established; in such respects, Farrell (1957) and Hicks (1970) set forth landmark arguments upon which the present contribution is meant to build.

Thus, consider the space $\Pi = (0,\infty)\times(0,\infty)$ of normalized prices π_x, π_y (i.e. prices to income) and the smooth mapping $\Delta : \Pi \to \mathcal{B}$ defined by (7) and (8).[9] Definitely, Δ is a bijection (diffeomorphism), and expansion paths on $\mathcal{B}$ are Δ-images of rays on Π (see for instance Cornes, 1992): the mapping Δ provides a parametrization of INE, and the straight expansion paths of homothetic models embody the property of homothetic

[9] Recall, Marshallian demand functions are functions of normalized prices, on account of the homogeneity of degree 0 of budget-constrained consumption problems.

Marshallian demands of mapping rays on Π onto rays on $\mathcal{B}$. Figure 6 provides a transparent representation of the bending of the expansion paths in Figure 3 as divergence from the CD limit $\varepsilon \to 0$.

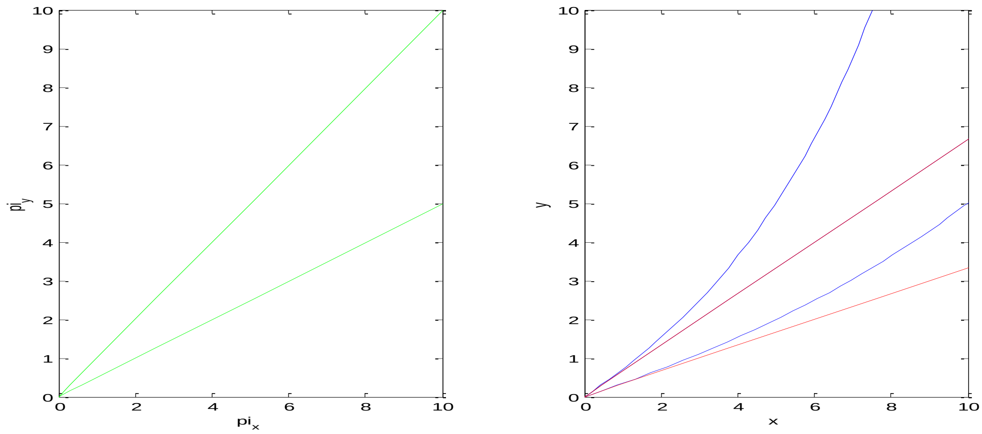

Figure 6 The expansion paths on bundle space (right, blue curves) pictured in Figure 3 as images of the rays $\pi_x = \pi_y$ and $\pi_x = 2\pi_y$ (left) in the space of normalized prices. Notice the tangency with the corresponding CD expansion paths (right, red lines) in the small income limit (eq. 9). For increasing income, the departure of INE from SCE is represented by bending of expansion paths towards the luxury axis.

Then, our first guest is a vector field which generates the group of homotheties in bundle space $\mathcal{B}$; indeed, the vector field

$$\mathbf{Z} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{17}$$

generates homotheties (SCE) in bundle space, and provides the geometric representation of the ODE system $\dot{x} = x$, $\dot{y} = y$, which admits the well known solution

$$x(t; x_0, y_0) = x_0\, e^t \quad , \quad y(t; x_0, y_0) = y_0\, e^t \tag{18}$$

for any t, which we can write $(x, y) = e^{t\,\mathbf{Z}}(x_0, y_0)$, so that $e^{t\,\mathrm{Z}}\left(e^{s\,\mathrm{Z}}(x_0, y_0)\right) = e^{(t+s)\mathbf{Z}}(x_0, y_0)$ for any t, s.

In homothetic models the integral curves of **Z** do superpose to expansion paths, so that one can employ **Z** (or functions thereof) in order to generate INE. We shall focus the departure from the scale invariance of homothetic models in order to tailor an index of luxury, and employ SCE as probes of such departure, thereby aligning with the perspective embodied by Shephard's distance $\mathcal{D}$. For our class (1) of preferences we can compute the distance $\mathcal{D}$ between a bundle $B \in \mathcal{B}$ and an indifference curve u by means of (18), with the initial condition $(x_0, y_0) \equiv J$ being the intersection of u with the ray through B, namely,

$$B = e^{t\,\mathbf{Z}} J \quad \Leftrightarrow \quad \mathcal{D}(B, u = U_{a,\varepsilon}(J)) = e^{t} \quad , \tag{19}$$

with unit distance (vanishing t) if and only if $B \in u$. In fact, for homothetic models, $\mathcal{D}$ is computed along expansion paths, and therefore can be employed as parametrization of INE. On the other hand, in a general problem expansion paths are not rays, and $\mathcal{D}$, in a precise sense, measures the departure from scale symmetry of the expansion paths. Along similar lines, our index (formula 32 below) defines a 'modulation' of SCE, and, as such, is connected with $\mathcal{D}$ according to (19).

As generator of scale transformations on $\mathcal{B}$, the vector field **Z** measures the degree of homogeneity h of a homogeneous function $\Phi(x,y)$ as

$$\mathbf{Z}(\Phi)(x, y) = h\,\Phi(x, y) \quad . \tag{20}$$

Z can be employed as well to measure the departure from scale symmetry: in fact, the generalized Euler equation (Chambers, 1988; McElroy, 1969)

$$\mathbf{Z}(U_{a,\varepsilon})(x, y) = (1 + \varepsilon(1-a)y)\, U_{a,\varepsilon}(x, y) = \chi_{a,\varepsilon}(y)\, U_{a,\varepsilon}(x, y) \tag{21}$$

provides a geometric representation of the significance of the deviation factor χ, namely, the *elasticity of scale* (for instance, Varian, 1992)

$$\frac{\mathbf{Z}(U_{a,\varepsilon})}{U_{a,\varepsilon}} = \chi_{a,\varepsilon}(y) \quad . \tag{22}$$

of the utility function. As is well known, elasticity of scale is a cardinal measure, and, as such, depends on the utility representation; still, the functional form of the deviation factor is intrinsic: for the general utility representation $\Gamma(U)$, the chain rule yields $\mathbf{Z}(\Gamma(U)) = \Gamma'(U)\,\mathbf{Z}(U)$, so that χ is unaffected by the representation, in that it enters the expression of elasticity of scale as a factor, with the given functional form $\chi_{a,\varepsilon}(y) \equiv 1 + (1-a)\varepsilon y$ which complies with the general expression of the elasticity of scale of generalized power production functions (see for instance Chambers, 1988).

A second guest of our approach is a vector field which generates SUE. Following Mantovi (2013), we proceed in two steps. The first step consists of introducing a vector field $\tilde{\boldsymbol{S}}$ tangent to indifference curves. Definitely, the components of the differential of U can be paired to the coordinate vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ so that the vector field

$$\tilde{\boldsymbol{S}} = \frac{\partial U}{\partial y}\frac{\partial}{\partial x} - \frac{\partial U}{\partial x}\frac{\partial}{\partial y} = e^{\varepsilon(1-a)y}\left((1-a)x^{a}y^{-a}(1+\varepsilon y)\frac{\partial}{\partial x} - ax^{a-1}y^{1-a}\frac{\partial}{\partial y} \right) \tag{23}$$

is tangent to indifference curves; in fact, one can easily check that $\tilde{\boldsymbol{S}}(U) = 0$. Such a vector field depends on the utility representation, and must be properly normalized in order to generate a flow whose parameter embodies the effects we are interested in. Thus, let us move to the second step of our procedure.

Let the ratio y/x be the coordinate on indifference curves by means of which we want to parametrize SUE; such a choice is pivotal for our SCE-SUE decompositions, in that such a parametrization of SUE is *adapted* to SCE. Compute the normalization function, i.e. the action of the vector field (23) on the function *y*/*x*,

$$\mathcal{N}_{\frac{y}{x}} = \tilde{\mathcal{S}}\left(\frac{y}{x}\right) = \left(\frac{\partial U}{\partial y}\frac{\partial}{\partial x} - \frac{\partial U}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{y}{x}\right)$$
$$= -e^{\varepsilon(1-a)y} x^{a-2} y^{1-a}(1+(1-a)\varepsilon y) = -\frac{1}{x^2}\chi_{a,\varepsilon}(y) U_{a,\varepsilon}(x,y) \quad . \tag{24}$$

As expected, we face the emergence of the deviation factor χ. Thus, the proper SUE vector field results in

$$\mathcal{S} \equiv \frac{\tilde{\mathcal{S}}}{\mathcal{N}_{\frac{y}{x}}} = -(1-a)\, x^2 y^{-1} \frac{1+\varepsilon y}{\chi_{a,\varepsilon}(y)} \frac{\partial}{\partial x} + \frac{ax}{\chi_{a,\varepsilon}(y)} \frac{\partial}{\partial y} \quad , \tag{25}$$

so that $\mathcal{S}\left(\frac{y}{x}\right) = 1$, with the correct CD limit for $\varepsilon \to 0$ (Mantovi, 2013). The vector field (25) is clearly independent of the utility representation, and its significance can be enlightened by the following application.

5 Elasticity of substitution

The introduction of elasticity of substitution is customarily attributed to Hicks (1932), whose aim was to sharpen the analysis of the substitutability of capital and labour. True, Hicks (1970) himself pointed out that J. Robinson (1933) “ought to have had the sole right” to such a concept, having been the first to identify its significance in terms of derivatives along isoquants. Elasticity of substitution is a unit-free measure of the substitutability of inputs, and plays a major role in the characterization of production functions (Chambers, 1988); true, once the number of factors exceeds 2, a number of corresponding elasticity concepts can be introduced, such as Hicks-, Allen- and Morishima-elasticity, and the precise characterization of the significance of such measures is far from trivial (see for instance Blackorby and Russell, 1989; Bertoletti, 2005). On consumers' side, as a characterization of the substitutability of consumption bundles, elasticity of substitution represents a key concept as well, as pointed out by Dixit and Stiglitz (1977).

Definitely, the vector field (25) sets a transparent framework for the computation of the elasticity of substitution σ of two consumption goods (inputs), in that it represents the derivative with respect to *y/x* along indifference curves (isoquants). Thus, the computation of σ for the preferences represented by (1) can be performed as

$$\sigma = \frac{d\ln\frac{y}{x}}{d\ln\frac{U_x}{U_y}}\Big|_{U=\text{constant}} = \frac{\frac{x}{y}}{\frac{U_y}{U_x}}\frac{1}{\boldsymbol{S}\left(\frac{U_x}{U_y}\right)} = \frac{1}{(1+\varepsilon y)\boldsymbol{S}\left(\frac{y}{x(1+\varepsilon y)}\right)}$$

$$= \frac{1}{1+\varepsilon y}\frac{1}{-(1-a)\,x^2y^{-1}\frac{1+\varepsilon y}{\chi_{a,\varepsilon}(y)}\frac{-y}{x^2(1+\varepsilon y)}+\frac{ax}{\chi_{a,\varepsilon}(y)}\frac{1}{x(1+\varepsilon y)^2}} = \frac{\chi_{a,\varepsilon}(y)}{(1-a)(1+\varepsilon y)+\frac{a}{1+\varepsilon y}} \tag{26}$$

with the correct unit CD limit for $\varepsilon \to 0$. The dependence on *y* is as expected: σ starts at 1 in the small *y* (CD) limit, displays an increasing phase, and then, after the peak, displays a decreasing phase which asymptotically approaches 1. Noticeably, σ turns out not to depend on *x*. In the author's vision, the transparency of the above algorithm for computing σ enlightens the effectiveness of our geometric approach.

As a further check of consistency, one can apply our algorithm to the CES (homothetic) functions $(Ax^{-\rho}+By^{-\rho})^{\frac{-1}{\rho}}$, with $\rho > -1$, and confirm that σ turns out indeed to be constant: being

$$MRS = \frac{\frac{\partial U}{\partial y}}{\frac{\partial U}{\partial x}} = \frac{B}{A}\left(\frac{x}{y}\right)^{1+\rho}, \tag{27}$$

by the chain rule we have

$$\mathcal{S}\left(\left(\frac{y}{x}\right)^{1+\rho}\right)=(1+\rho)\left(\frac{y}{x}\right)^{\rho} \tag{28}$$

for any vector field $\mathcal{S}$ normalized as in (24), and then for the proper SUE vector field defined by a CES function. We thereby recover the constant values $\sigma = 1/(1+\rho) \in (0,\infty)$ (Arrow et al. 1961), which embody the scale symmetry of CES[10] models.

One can compare the generality of our geometric approach with the computations of elasticity of substitution in Dixit and Stiglitz (1977) and Blackorby and Russell (1989), which capitalize on the scale symmetry of homothetic models, in which MRS are constant along rays and can therefore can be expressed as functions of coordinate ratios. That is not the case for our model, and in general for any non homothetic problem. A well known formula for the elasticity of substitution in 2 variables is set forth by Allen (1966, section 13.7); with respect to such a formula, one can appreciate the transparency of the algorithm (26).

6 The index

Finally, we are in a position to compute the Lie bracket $[\mathcal{S},\mathbf{Z}]$, and thereby introduce an index of luxury. We may proceed straightforwardly by performing the standard algebra (for instance, Arnold, 1989, for which *Poisson bracket* stands for *Lie bracket*). True, an educated guess about the form of $[\mathcal{S},\mathbf{Z}]$ goes as follows. First, we expect such a vector field to smoothly vanish in the $\varepsilon \to 0$ limit, on account of Mantovi (2013, Proposition 2). Second, we expect such a vector field to be radial, since SUE connect rays, and therefore it is SCE which drive the departure from commutation. Evidently, we expect the deviation factor to enter the definition of $[\mathcal{S},\mathbf{Z}]$. To sum up, we expect the vector field $[\mathcal{S},\mathbf{Z}]$ to be of the form $\varepsilon\,\psi(\chi)\,\mathbf{Z}$. We are in a position to employ the standard algebra in order to check the properness of our argument, and find

[10] In such respects, the relevance of CES and CD preferences as sufficient conditions for the existence of a representative agent has long been established; see for instance Acemoglu (2008).

$$[\boldsymbol{\mathcal{S}},\mathbf{Z}]_x = \mathcal{S}_x \frac{\partial Z_x}{\partial x} + \mathcal{S}_y \frac{\partial Z_x}{\partial y} - Z_x \frac{\partial \mathcal{S}_x}{\partial x} - Z_y \frac{\mathcal{S}_x}{\partial y}$$

$$\begin{aligned} &= -(1-a)x^2 y^{-1} \frac{1+\varepsilon y}{\chi(y)} \cdot 1 + ax \frac{1}{\chi(y)} \cdot 0 \\ &- x(a-1)2xy^{-1} \frac{1+\varepsilon y}{\chi(y)} - y(1-a)x^2 y^{-2} \frac{1+\varepsilon y}{\chi(y)} + y(1-a)x^2 y^{-1} \frac{d}{dy}\left(\frac{1+\varepsilon y}{\chi(y)}\right) \\ &= (1-a)x^2 \frac{d}{dy}\left(\frac{1+\varepsilon y}{\chi(y)}\right) \end{aligned} \tag{29}$$

for the *x* component, and

$$[\boldsymbol{\mathcal{S}},\mathbf{Z}]_y = \mathcal{S}_x \frac{\partial Z_y}{\partial x} + \mathcal{S}_y \frac{\partial Z_y}{\partial y} - Z_x \frac{\partial \mathcal{S}_y}{\partial x} - Z_y \frac{\partial \mathcal{S}_y}{\partial y}$$

$$\begin{aligned} &= -(1-a)x^2 y^{-1} \frac{1+\varepsilon y}{\chi(y)} \cdot 0 + ax \frac{1}{\chi(y)} \cdot 1 - xa \frac{1}{\chi(y)} - y \cdot 0 \frac{1}{\chi(y)} - yax \frac{d}{dy} \frac{1}{\chi(y)} \\ &= -yax \frac{d}{dy} \frac{1}{\chi(y)} \end{aligned} \tag{30}$$

for the *y* component. Definitely, the vector field whose coordinate components are given by (29) and (30) is radial, as can be checked by computing the following derivatives:

$$\frac{d}{dy}\left(\frac{1+\varepsilon y}{\chi(y)}\right) = \frac{a\varepsilon}{(1+(1-a)\varepsilon y)^2} \quad , \quad \frac{d}{dy}\frac{1}{\chi(y)} = \frac{(a-1)\varepsilon}{(1+(1-a)\varepsilon y)^2} \quad . \tag{31}$$

Thus, the ratio of (30) to (29) results in y/x, so that the vector field [$\mathcal{S}$,**Z**] is a multiple of ε in the radial direction, thereby confirming our educated guess. As expected, SCE generated by **Z** do not commute with SUE generated by the vector field (25), and their commutator results in effects along rays generated by the vector field (independent of the utility representation)

$$[\boldsymbol{S},\mathbf{Z}] = \varepsilon \frac{a(1-a)}{(1+(1-a)\varepsilon y)^2}\left(x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} \right) = \varepsilon\, a(1-a) \frac{x}{\chi_{a,\varepsilon}{}^2(y)} \mathbf{Z} \equiv L(x, y; a, \varepsilon)\, \mathbf{Z}$$

(32)

with the correct vanishing limit for $\varepsilon \to 0$. *L* is an index of luxury which, for any function belonging to the class (1), defines a function on bundle space meant to 'modulate' the effect of the vector field **Z** so as obtain the proper generator of the noncommutativity of SUE and SCE. The function *L* increases with *x* and decreases with *y*, on account of the link with Shephard's distance (next section). As a consequence of the non vanishing of (32), the flows of the vector fields $\boldsymbol{S}$ and **Z** do not commute, i.e. in general

$$e^{s\boldsymbol{S}} e^{t\mathbf{Z}} A \neq e^{t\mathbf{Z}} e^{s\boldsymbol{S}} A \tag{33}$$

for initial condition *A* and flow parameters *t*, *s*. In order to bridge (32) and (33), let us employ the following standard recipe in the coordinate approach to differential geometry.

Consider infinitesimal SCE and SUE and their coordinate representation in terms of the components of the vector fields. Such components, recall, define the corresponding ODE system; they are *infinitesimal generators* of the flows. Thus, consider the following infinitesimal path *ATB* in bundle space built out of a SCE followed by a SUE:

$$x_T = y_T = y_A + t\, \mathbf{Z}_y(A) + o(t) \quad , \quad x_B = 2y_B = 2(y_T + s\, \boldsymbol{S}_y(T)) + o(s) \quad , \tag{34}$$

and then Taylor expand $\boldsymbol{S}_y(T)$ about *A* in order to obtain the approximation

$$y_B \cong y_A + t\, Z_y(A) + s\, \boldsymbol{S}_y(A) + ts \frac{\partial \boldsymbol{S}_y}{\partial x}(A) Z_x(A) + ts \frac{\partial \boldsymbol{S}_y}{\partial y}(A) Z_y(A) \quad . \tag{35}$$

Consider then the infinitesimal path *ADF* in bundle space built out of a SUE followed by a SCE:

$$x_D = 2y_D = 2(y_A + s\,\mathcal{S}_y(A)) + o(s)\ ,\ x_F = 2y_F = 2(y_D + t\,\mathrm{Z}_y(D)) + o(t)\ , \quad (36)$$

and Taylor expand $Z_y(D)$ about A in order to obtain the approximation

$$y_F \cong y_A + t\,Z_y(A) + s\,\mathcal{S}_y(A) + ts\frac{\partial Z_y}{\partial x}(A)\mathcal{S}_x(A) + ts\frac{\partial Z_y}{\partial y}(A)\mathcal{S}_y(A) \quad (37)$$

Then, take the difference $x_B - x_F$ and the *t*- and *s*- derivatives (Arnold, 1989, p. 210) and recover the algebra represented in (30). A corresponding argument enables us to derive the algebra in (29). Thus, the non vanishing Lie bracket (32) measures the distance between the final points of the two infinitesimal paths, i.e. the two paths do not define a loop,[11] and the infinitesimal radial 'missing' path is generated by the vector field (32).

In order to enlighten such a differential geometric result, in the following section we address the link with Shephard's distance, so as to deepen the conceptual relevance of our approach.

7 Deviation and distance

The Lie bracket (32) is a differential measure of the noncommutativity of the flows of $\mathcal{S}$ and **Z**. In order to apply such a measure to finite effects we need integrate (32) along finite paths. The following argument is meant to pin down the essentials of such a process in terms of Shephard's distance. Again, fix the consumer a = 0.75, ε = 0.4, and consider the following finite paths built out of SUE and SCE in different order, as pictured in Figure 7.

The first path is built as follows. Start with the bundle A = (2,2), with $U(A) = 2e^{0.2}$, and apply a SCE leading to the bundle T = (4,4), with $U(T) = 4e^{0.4}$. Such a SCE corresponds to the flow of **Z** starting at A with parameter ln 2, i.e. $T = (4,4) = e^{(\ln 2)\mathbf{Z}}A$. Then, on account of (19),

$$\mathcal{D}(T, U(A)) = e^{\ln 2} = 2 \quad . \quad (38)$$

[11] We refer to the literature on differential geometry for the generality of such an argument.

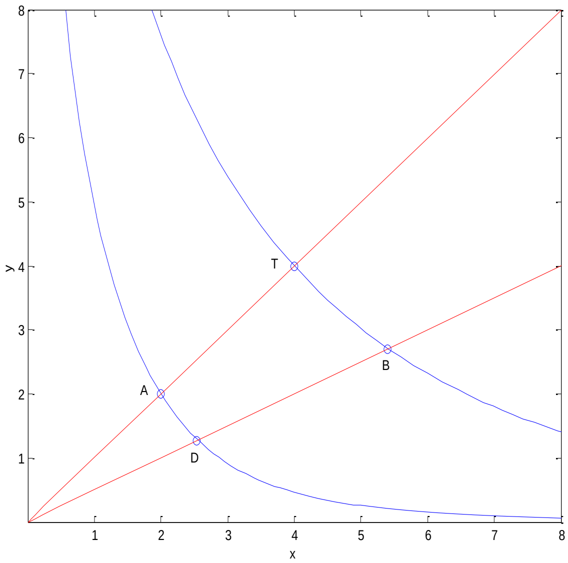

Figure 7 Indifference (blue) curves for levels $U(2,2) = U(A) = U(D) = 2e^{0.2}$ and $U(4,4) = U(T) = U(B) = 4e^{0.4}$, a = 0.75 and ε = 0.4. Scale effects act along the (red) rays $x = y$ and $x = 2y$.

Then, apply a SUE (acting along the indifference curve) leading to the bundle B belonging the ray $x = 2\,y$. The components of $B = e^{-0.5\,\boldsymbol{S}}T$ are given by $x_B = g^{-1}(4e^{0.4})$, roughly 5.413, and $y_B = 0.5\, x_B$, being

$$g(s) = 2^{-0.25} s\, e^{s/20} \quad . \tag{39}$$

We can represent the distance $\mathcal{D}(B, U(A))$ by commuting the previous effects.

The second path is built as follows. Again, start with the bundle A = (2,2) and apply the SUE leading to the bundle $D = e^{-0.5\,\boldsymbol{S}}A$ belonging to the ray $x = 2y$. The components of D are given by $x_D = g^{-1}(2e^{0.2})$, roughly 2.556, and $y_D = 0.5\, x_D$. Then, apply the SCE leading to the bundle B; such a SCE corresponds to the flow of **Z** starting at D with parameter $\bar{t}$, i.e. $B = e^{\bar{t}\,\mathbf{Z}}D$. Then, Shephard's distance $\mathcal{D}(B, U(D))$ can be represented as

$$\mathcal{D}(B,U(A)) = \mathcal{D}(B,U(D)) = e^{\bar{t}} = \frac{x_B}{x_D} = \frac{y_B}{y_D} \quad . \tag{40}$$

Basically, the loop ATBDA enables us to frame the noncommutativity of SCE and SUE in terms of the difference between (40) and (38): the two paths ATB and ADB are built so as to lead to the same final point *B*, and to do so they must accommodate SCE of different 'size', i.e. Shephard's distance. It is this difference $e^{\bar{t}} - e^{\ln 2}$ by means of which we want to enlighten the differential measure (32). Such a difference is built out of the solutions (18) for the flow of **Z**: along any ray we can expand bundles by the group of homotheties (18), and then map such expansion into a corresponding expansion of utility levels. Evidently, the ray *x* = 2*y* contains less relative luxury with respect to the ray *x* = *y*, so that we expect the expansion factor to be larger along the ray *x* = 2*y* in order to obtain the same expansion of utility. In fact, by means of the function *g* we can assess the precise extent to which $\bar{t}$ exceeds ln 2 (equivalently, $x_B > 2\, x_D$), and then, to what extent Shephard's distance depends on the size of SUE connecting our rays.

Now, consider the limit in which *T* approaches *A* along the ray, and correspondingly for *B* and *D*, and the SUE are infinitesimal. Then, one reduces to the infinitesimal effects represented by (35) and (37), in which the finite distances represented in (38) and (40) reduce to infinitesimal distances along infinitesimally nearby rays. We thereby succeed in providing an intuitive link between *L* and $\mathcal{D}$, as embodied by the expressions

$$
\begin{aligned}
x_F - x_B &\cong t\,[\boldsymbol{S}, \mathrm{Z}]_x(A) = t\, L(A)\, x_A = t\, \varepsilon\, a(1-a)\frac{x_A^2}{\chi^2_{a,\varepsilon}(y_A)} \\
y_F - y_B &\cong t\,[\boldsymbol{S}, \mathrm{Z}]_y(A) = t\, L(A)\, y_A = t\, \varepsilon\, a(1-a)\frac{x_A y_A}{\chi^2_{a,\varepsilon}(y_A)}
\end{aligned}
\qquad (41)
$$

which define a transparent representation of the effects of ε and χ as drivers of departure from the CD limit in which INE and SCE are equivalent, and do commute with SUE. To sum up, the following proposition distils the essence of our analysis.

Proposition 2 *The Lie bracket* [$\boldsymbol{S}$,**Z**] = *L***Z** *enables us to introduce an index of luxury for our model as a function L* (*x,y*) *on bundle space. The index L vanishes in the CD limit, and provides a scaling measure of the noncommutativity of infinitesimal scale effects and substitution effects, thereby establishing a close link with Shephard's distance.*

8 Perspectives

It was the aim of the present contribution to argue about the generality and tractability of a 2-parameter class of ordinal utility functions over a pair of goods; such a class contains Cobb-Douglas functions as no-luxury limit. We have succeeded in establishing explicit closed form solutions for Marshallian demand functions, expansion paths, Engel curves, income elasticity of demand, saturation levels, inverse demand, elasticity of scale and elasticity of substitution. For positive ε functions (1) embody general traits of preferences for luxury: expansion paths bend towards the luxury axis and display asymptotes parallel to such an axis, thereby representing satiation of necessity. By its flexibility, the model (1) is well suited to the empirical analysis of consumption, to the extent that smooth reduction to Cobb-Douglas models represents a sound benchmark. In such respects, the factorization of the 'luxury effect' in the function $F(y) = \exp(\varepsilon(1-a)\,y)$ is crucial (needless to say, with respect to different utility representations such a factorization turns into different analytical mechanisms).

Then, building on the geometric approach set forth by Mantovi (2013), we have succeeded in introducing an index of luxury L as a function on bundle space, i.e. a function of the endogenous variables of the problem. Such an index is independent of the utility representation, and is built out of an algorithm which does not call for the solution of the consumption problem, and contains no more than standard differentiations. We have succeeded in representing our index in terms of Shephard's distance, thereby enlightening the theoretical relevance of our geometric approach to scale and substitution effects; in connection with our 'map of luxury' (1), we thereby deepen the benchmark relevance of homothetic models in the representation of preferences. On top of that, our differential geometric approach enables us to capitalize on a well established body of results, by means of which to set free from the straitjacket of comparative statics. The effectiveness of our approach is witnessed for instance by the transparency of our computation of elasticity of substitution.

With respect to the standard indicator of luxury, i.e. income elasticity of demand exceeding 1, our index provides a cogent representation of the geometrical properties of expansion paths. One may interpret ε as a measure of the 'intensity' of luxury, and our index L as a geometric characterization of deviation from the Cobb-Douglas limit. By its flexibility, our approach may enlighten the role of homotheticity as sufficient condition for the existence of acceptable welfare indicators (Chipman and Moore, 1980). By its analytical tractability, our class (1) of functions may define 'workhorse' models of production and utility in growth theories.[12] Exchange via the Edgeworth box may represent another pregnant application of our model, for instance with respect to the informational problems tailored by mechanism design (see for instance Hurwicz and Reiter, 2006).

Application of our framework to production analysis may pertain in first instance to the non equivalence of standard and reversed decompositions of productive efficiency as tailored by Bogetoft et al. (2006). Such Authors establish that the decomposition of overall efficiency into technical and allocative efficiency depends on the order with which such measures are computed, unless the technology is ray-homothetic or input-homothetic. Mantovi (2013) provides a theoretical foundation for such an occurrence in single-output settings. Definitely, one may employ the class (1) of production functions in order to derive general qualitative properties about the relevance of the order with which technical and allocative efficiency are computed; in addition, fairly general quantitative properties may emerge, once the Cobb-Douglas limit[13] represents a sound benchmark of analysis. True, in the author's vision, the relevance of the Lie bracket (32) lies in first instance in providing a line of thought for deepening the significance of Shephard's distance, which, recall, provides a complete representation of preferences. In fact, due to the monotonicity in y of both deviation factor and Shephard's distance in our model, (32) is equivalent to (1).

According to McFadden (2013), the microeconomics of choice as constrained utility maximization is "largely a finished subject," on account of a number of promising behavioral directions of progresses.[14] In fact, our geometric approach to duality is meant to employ the theory of dynamical systems in order to complement standard comparative statics analysis, and thereby deepen the analysis of partial equilibrium on

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[12] Acemoglu (2008, Section 8.7) argues about the relevance of CES intertemporal consumption for the existence of a unique path of balanced growth. Possibly, our model of preferences may help enlighten such an issue.

[13] Compare the class of production functions discussed by Bertoletti and Rampa (2013), for which an analogous benchmark limit does not seem to occur.

[14] In such respects, Gintis (2009) envisages a methodological unification of behavioral sciences in which the bounds of reason reflect social dimensions of choice.

both conceptual and technical grounds. In such respects, constrained utility maximization may prove useful in the inquiry about behavioral models of choice.

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