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DIFFERENTIAL DUALITY

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Parole chiave : Duality; homotheties; expansion vector field; distance function; expenditure function; conjugation, homotheticity; Lie bracket.

Differential duality

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Abstract. *A framework for duality theory is set forth, based on the model-independent role of homotheties. An expansion vector field is introduced as global generator of pure expansion effects, in terms of which to deepen the differential aspects of preference symmetries, fundamental identities and conjugation in connection with the benchmark nature of homothetic models. Potential lines of progress are envisaged.*

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1. Introduction

During the second half of the twentieth century the theory of duality has represented a major line of progress for the microeconomics of production and consumption. By the seventies, the equivalence of primal and dual representations of problems was undisputed as a firm ground upon which to build microeconomic analysis. As well undisputed is the benchmark nature of homothetic models, as witnessed by the role of such models in authoritative expositions of duality theory (for instance, Diewert, 1982; Chambers, 1988; Cornes, 1992). Still, the differential aspects of scale symmetry may be underexploited as yet, and tailor lines of progress for the theory of duality. In fact, it is the aim of the present contribution to set forth a differential geometric approach to preference symmetries, fundamental identities and conjugation, in connection with the benchmark properties of homothetic models, and thereby argue about the effectiveness of such an approach. For the sake of definiteness, we shall focus the problem of the consumer, taking for granted the isomorphism with the problem of the single output producer.

The vision underlying the present contribution posits that the well behavior of expansion paths on primal space can be consistently represented by the existence of an *expansion* vector field, whose integral curves provide global parametrizations of income (wealth) effects ("INE" henceforth). Such an expansion vector field is not a 'novelty'; for instance, a close correspondent is employed by Smale (1982, p. 343) in the global analysis of equilibrium. A novelty may be represented by the recognition of the effectiveness of such a vector field in the investigation of issues which lie beyond the reach of comparative statics.

The structure of duality theory is typically recognized as the network of 'links' between primal and dual representations of the fundamental properties of a system, for which Blackorby et al. (1978, Appendix) set forth a landmark analysis. Standard approaches seem to posit that duality theory is meant to focus the *model-dependent* objects of the theory (preferences, expenditure functions, distance functions, etc.). Definitely, a proper introduction of the expansion vector field forces us to 'strengthen' the *model-independent* structure of the theory on differential geometric grounds, and identify three levels of duality.

A first level, call it *pairing duality*, pertains to the very nature of dual variables. There exists a bilinear symmetric operation which takes a pair of dual variables and yields a real number. In such respects, prices and quantities are dual in quite the same sense that the elements of dual linear spaces are dual (with the caveat that our economic dual spaces do not coincide with their linear hull).

A second level of the theory may be called *scale duality* and pertains to the actions on the constraint structure of the group of homotheties on both dual spaces. Recall, expenditure minimization at fixed prices is easily envisioned in terms of expanding (or shrinking) budget constraints up to identifying the least expenditure that guarantees the reference utility level. The duality of the model-independent scale transformations of the constraint structure guarantees the equivalence of utility maximization and expenditure minimization problems, which lies at the foundations of duality theory (for instance, Mas-Colell et al., 1995).

A third model-dependent level of the theory, call it *demand duality*, deals with specific properties of preferences and demand. We shall assume that Marshallian demand defines a smooth bijection between dual spaces, and then consider the induced mappings on vector fields (scaling and expansion vector fields) and 1-forms (representing demand). The notions of *pull-back* and *push-forward* (Abraham and Marsden, 1987) are the natural instruments by means of which to implement the perfect symmetry between the dual formulations of producers' and consumers' problems.

Differential geometric methods have long entered microeconomic analysis (see for instance Debreu, 1976). Recently, Tyson (2013) employs vector fields on primal space in order to characterize symmetries of preferences in terms of PDEs for direct utility functions. Our framework is well suited to embed such an approach to continuous symmetries, and we shall fix the simple connection between the expansion vector field and a symmetry vector field (section 5), if any. In fact, our approach does fit more general settings in which the vanishing of the Lie derivative with respect to $\mathbf{Y}$ of a tensor field $\mathbf{t}$ is the condition for $\mathbf{t}$ to be constant along the flow of the vector field $\mathbf{Y}$ (Abraham and Marsden, 1987, chapter 2). Then, the 'reciprocity' condition for a pair of vector fields to be constant along each other's flows can be expressed in terms of their Lie bracket. The vector fields, and the associated flows, are then said to commute. Along such lines, Mantovi (2013) establishes that suitably parametrized expansion and substitution effects do commute for homothetic problems, and thereby deepens the link between standard and reversed decompositions of overall (Farrell) efficiency as tailored by Bogetoft et al. (2006).

The plan of the rest of the paper is as follows. In sections 2 and 3 we address pairing duality and scale duality respectively. In section 4 we introduce the expansion vector field. The connection with preference symmetries is established in section 5. The fundamental identities of demand are discussed in section 6, together with conjugation. In section 7 we address the benchmark nature of homothetic models in terms of the commutation of INE and scale effects. A final section is meant to tailor potential lines of progress.

We refer to Abraham and Marsden (1987) and Spivak (1999) for the geometric notions we shall be dealing with (manifolds, diffeomorphisms, vector fields, flows, Lie brackets, 1-forms). Admittedly, the following discussion makes no justice of the relevance of duality in the development of microeconomic theory, the aim of the paper being essentially methodological. In such respects, we shall try and avoid employing economic insights in the derivation of results, with the aim of sharpening the *structure* of the theory. For instance, convexity shall not enter explicitly the following discussion, being subsumed in the well behavior of preferences. We shall write "d-h" for the property of a function of being homogeneous of degree d in definite variables.

2. Pairing duality

Our geometric setup is as follows. Let the primal manifold $\mathcal{B}$ be defined by the strictly positive orthant[1] $(0,\infty)\times\ldots\times(0,\infty) \subset \mathbb{R}^n$ endowed with the natural C^∞ (*smooth*) differentiable structure; let $\mathcal{B}$ represent the consumption set of our price-taking consumer. The natural coordinates $\mathbf{q}$ of the global chart $(0,\infty)\times\ldots\times(0,\infty)$ of $\mathcal{B}$ represent a preferred coordinate system: despite the infinite coordinate systems accommodated by the maximal atlas of $\mathcal{B}$, economic arguments are meant to deal with quantities of well defined infinitely divisible goods, which the coordinates $\mathbf{q}$ are meant to represent.

Prices $\mathbf{P}$ and income (wealth) I are the exogenous variables of our problems. The space Δ of such variables is $(n+1)$-dimensional, and Marshallian demand is a correspondence $\Delta \to \mathcal{B}$. In fact, Marshallian demand is 0-h on Δ, and we are in a position to break such a symmetry and thereby define Marshallian demand as a correspondence from a dual n-dimensional space to $\mathcal{B}$. As is well known, one can for instance elect a commodity as numeraire, and thereby define relative prices. Definitely, we shall break 0-h via *normalized* prices, i.e. prices-to-income (for instance, Cornes, 1992). Thus, let $\mathcal{A}$ represent the space of *positive* normalized prices (rule out free goods), and let Marshallian demand be a correspondence $\mathcal{A} \to \mathcal{B}$. Write $\pi : \Delta \to \mathcal{A}$ for the projection $\pi(\mathbf{P}, I) = \mathbf{P}/I \equiv \mathbf{p}$.

If we consider $\mathcal{B}$ as a subset of a n-dimensional real linear space Λ (true, consumption bundles can be summed and scaled up as long as the result belongs to $\mathcal{B}$), we can define the *dual* manifold $\mathcal{A}$ as the strictly positive orthant of the dual Λ^* endowed with the natural differentiable structure, and thereby inherit the pairing $\mathbf{pq}$ between elements of the dual linear spaces. Thus, elements of $\mathcal{A}$ define linear forms on $\mathcal{B}$ and viceversa, in close correspondence with the natural isomorphism between a finite dimensional linear space and the dual to its dual (see for instance Spivak, 1999, p. 108). This is a first level of pairing duality, whose matrix representation reads

$$\mathbf{pq} \equiv \sum_{k=1}^{n} p_k q^k \equiv \begin{pmatrix} p_1 & \ldots & p_n \end{pmatrix} \begin{pmatrix} q^1 \\ \vdots \\ q^n \end{pmatrix} \tag{2.1}$$

Gorman (1976, p. 238) points out the relevance of the *symmetry* of such a pairing.

Elements of $\mathcal{A}$ represent the normalized prices of our consumption goods, and therefore any change in the coordinates on $\mathcal{B}$ calls for a suitable change of the coordinates on $\mathcal{A}$ such that the pairing $\mathbf{pq}$ is well defined, i.e. invariant with respect to such transformations[2].

The condition $1 = \mathbf{pq}$ enables us to define the constraint structure on our dual spaces. On the one hand, given $\mathbf{q}\in\mathcal{B}$, such a condition defines the locus (simplex) $\alpha(\mathbf{q}) \equiv \{\mathbf{p}\in\mathcal{A} : 1 = \mathbf{pq}\}$ of normalized prices for which $\mathbf{q}$ entails unit normalized expenditure. On the other hand, given $\mathbf{p}\in\mathcal{A}$, the locus $\beta(\mathbf{p}) \equiv \{\mathbf{q}\in\mathbf{B} : 1 = \mathbf{pq}\}$ (a simplex) represents the boundary of the budget constraint, i.e. the set of bundles which can be bought at the prevailing prices assuming all income is spent. Such constraints define infinite foliations of $\mathcal{A}$ and $\mathcal{B}$ that can be parametrized by expansion.

Definitely, not only $\mathcal{A}$ and $\mathcal{B}$ are dual, they are diffeomorphic. Recall, a C^k diffeomorphism is a C^k bijection between manifolds (then necessarily of the same dimension). It is not difficult to convince oneself of the existence of a smooth diffeomorphism between $\mathcal{A}$ and $\mathcal{B}$ by employing the natural global charts. We are thus in a position to address the duality mapping $\mathcal{B} \to \mathcal{A}$ (given by inverse Marshallian demand) in terms of the powerful toolkit of manifolds and mappings, for which we recall the following basic notions.

A diffeomorphism $\phi : \mathcal{A} \to \mathcal{B}$ enables one to *push-forward* any smooth tensor field on $\mathcal{A}$ and thereby define a smooth tensor field of the same type on $\mathcal{B}$. The simplest case is the push-forward of a function (a tensor filed of type 0-0) $g : \mathcal{A} \to \mathbb{R}$, which is defined as $\phi_* g : \mathcal{B} \to \mathbb{R}$, $\phi_* g\ (b) \equiv g\ (\phi^{-1}(a))$, for any

[1] We do not consider the boundary of the orthant, in that we do not want nonnegativity constraints to be binding. Compare Cornes (1992, p. 35).

[2] Such coordinate changes may represent 'combined' goods and prices, for instance, characteristics and hedonic prices. Recall, in general, elements (points) of a manifold do not admit linear forms defined on them intrinsically.

$b = \phi^{-1}(a) \in \mathcal{B}$. Corresponding definitions apply to tensor fields of arbitrary type (a significant example of tensor field of rank 2 on $\mathcal{A}$ is given by the matrix of substitution effects).

A smooth surjective function $f : \mathcal{A} \to \mathcal{B}$ enables one to *pull-back* any smooth tensor field on $\mathcal{B}$ and thereby define a smooth tensor field of the same type on $\mathcal{A}$. The simplest case is the pull-back of a function $h : \mathcal{B} \to \mathbb{R}$ defined as $f^*h : \mathcal{A} \to \mathbb{R}$, $f^*h\,(a) \equiv h\,(f\,(a))$ for any $b = f\,(a) \in \mathcal{B}$. Corresponding definitions apply to tensor fields of arbitrary type. Evidently, being f a diffeomorphism, the pull-back f^* is the inverse of the push-forward $f^{-1}{}_*$. One can define an indirect utility function as the pull-back of a direct utility function via Marshallian demand, provided the following assumption holds.

Assumption 2.1. *Let preferences be smooth and satisfy nonsatiation* (*strong monotonicity*) *and strictly convexity* (compare Cornes, 1992, chapter 2) *so that Marshallian demand* $\varphi : \mathcal{A} \to \mathcal{B}$ *is a diffeomorphism*.

Assumption 2.1 is meant to represent well behavior as smoothness and uniqueness of optimal consumption, and rule out for instance nonconvexities. It is satisfied for instance by CES (and then Cobb-Douglas) preferences. By Assumption 2.1 we are guaranteed the possibility to pull-back a direct utility function and define an indirect utility function, as well as push-forward an indirect utility function and obtain direct utility, and then map dual indifference sets one onto another. Then, being Marshallian demand a diffeomorphism, we are guaranteed that expansion paths do foliate $\mathcal{B}$ smoothly, thereby paving the way to the representation of expansion paths and INE in terms of the flow of a smooth vector field (Proposition 4.1 below).

The fundamental role of diffeomorphisms in differential geometry has been long established: diffeomorphic manifolds are in a sense the same manifold, in that any class of geometric objects on one manifold can be pushed forward on the other manifold and retain the same properties (see for instance Abraham and Marsden, 1987, Proposition 1.7.17). Definitely, normalized prices enable us to break 0-h of Marshallian demand and obtain dual diffeomorphic manifolds $\mathcal{A}$ and $\mathcal{B}$ upon which scaling vector fields have the same analytic form, and thereby get the chance to exploit the full power of diffeomorphic descriptions.

Recall some notation. A vector field on the manifold $\mathcal{M}$ of dimension n is a section of the tangent bundle $T\mathcal{M}$. Within each chart, the coordinate vector fields $\frac{\partial}{\partial x^k}$ (k = 1, ... , n) define a basis of each tangent space to $\mathcal{M}$, so that a vector field $\mathbf{Y}$ on $\mathcal{M}$ can be written

$$\mathbf{Y}(x) = \sum_{k=1}^{n} Y^k(x)\frac{\partial}{\partial x^k} \tag{2.2}$$

and its push-forward $\phi_*\mathbf{Y}$ as

$$\begin{aligned}\phi_*\mathbf{Y}(x) &= \sum_{j,k=1}^{n} \left(Y^k(x)\frac{\partial \phi^j}{\partial x^k}\right)\left(\frac{\partial x^k}{\partial \phi^j}\frac{\partial}{\partial x^k}\right) \\ &\equiv \sum_{k=1}^{n} Y^k(\phi)\frac{\partial}{\partial \phi^k}\end{aligned} \tag{2.3}$$

A 1-form on $\mathcal{M}$ is a section of the cotangent bundle $T^*\mathcal{M}$. Within each chart, the coordinate 1-forms dx^j (j = 1, ... , n) define a basis of each cotangent space to $\mathcal{M}$, so that a 1-form $\boldsymbol{\omega}$ on $\mathcal{M}$ can be written

$$\boldsymbol{\omega}(x) = \sum_{j=1}^{n} \omega_j(x)dx^j \tag{2.4}$$

Coordinate vector fields and 1-forms are *dual*, in that, by definition,

$$dx^j\left(\frac{\partial}{\partial x^k}\right) \equiv \delta_k^j \tag{2.5}$$

being δ_k^j (=1 for $k = j$, 0 else) the components of the Kronecker tensor. Then, the pairing between a 1-form and a vector field results in a scalar function whose coordinate representation, on account of (2.5), reads

$$\boldsymbol{\omega}(\mathbf{Y})(x) = \sum_{j=1}^{n} \omega_j(x)Y^j(x) \tag{2.6}$$

On algebraic grounds, (2.5) represents the standard duality between elements of dual bases; on geometric grounds, (2.5) represents a *differential* level of duality. The economic interpretation of (2.1) is that the cost of a consumption bundle is the linear combination of prices and quantities, each price contributing only to the cost of consuming one good. The differential consequence of such an interpretation is represented by (2.5): an infinitesimal variation of the normalized price p_j affects only the normalized cost of varying consumption of good j. We thereby envision two levels of pairing duality, a *base* level represented by (2.1), and a *tangent* level represented by (2.5). Such a distinction enables a better appreciation of the following results, whose consistency rests on

Property 2.2. *The global coordinates* **p** *on* $\mathcal{A}$ *induce natural coordinates on each tangent space to* $\mathcal{A}$*. Correspondingly, the global coordinates* **q** *on* $\mathcal{B}$ *induce natural coordinates on each tangent space to* $\mathcal{B}$.

Proof. Beginners in differential geometry soon become acquainted with such a property for finite dimensional linear spaces (for instance Spivak, 1999, chapter 3). In fact, our finite dimensional dual spaces $\mathcal{A}$ and $\mathcal{B}$ are not linear spaces, since they do not coincide with their linear hull. Still, it is not difficult to adapt standard arguments in order to convince oneself that Property 2.2 holds on our dual spaces, since, given any $\mathbf{q}\in\mathcal{B}$, the points $\mathbf{q} + t\,\mathbf{v}$ belong to $\mathcal{B}$ for any $\mathbf{v}\in\mathcal{B}$ for small enough $|\,t\,|$, and correspondingly for $\mathcal{A}$.

Property 2.2 sets the status of the tangent level of duality (see Remark 3.2 below).

3. Scale duality

It is well known that the action of the group of homotheties (scale transformations) of $\mathbb{R}^n$ can be realized as a flow, by means of which we can strengthen the absolute level of duality theory. In fact, rays on $\mathcal{A}$ have been effectively employed in the representation of INE (Cornes, 1992, chapter 3).

Recall, the role of scale transformations in shaping the inquiry about index numbers, and in fact in shaping the structure itself of microeconomics, has long been recognized (for instance, Malmquist, 1953; Shephard, 1953). We shall employ scale transformations in order to parametrize INE in *ratio* form, thereby departing from the standard linear parametrization employed in Slutsky equations, and aligning with the philosophy embodied by the distance function.

Definitely, the integral curves of the vector field

$$\Xi = \sum_{j=1}^{n} p_j \frac{\partial}{\partial p_j} \qquad (3.1)$$

on $\mathcal{A}$ do span all of the rays of $\mathcal{A}$, as confirmed by the solutions (with $j = 1,...,n$, $-\infty < s < +\infty$), for any initial condition $\mathbf{p}(0)$,

$$p_j(s) = p_j(0)\,e^s \qquad (3.2)$$

to the initial value problems defined by the first order ODE system $\frac{d}{ds} = \Xi$; write $\mathbf{p}(s) = e^{s\Xi}\,\mathbf{p}(0)$ as a compact notation for the n maps (3.2). Notice that the dynamics (3.2) is decoupled, i.e. each price evolves independently of the others.

The flow generated by the curves (3.2) provides a consistent definition of INE: given any $\mathbf{p}\in\mathcal{A}$, the point $e^{-s}\mathbf{p}$ uniquely identifies the INE in which prices are scaled by the factor e^{-s} at fixed income, or, equivalently, income is scaled by the factor e^{s} at fixed prices. The exponential notation is well adapted to the composition of effects, and one can write $e^{-s}(e^{-t}\,\mathbf{p}) = e^{-(s+t)}\,\mathbf{p}$ for the composition law for the flow of a vector field (Abraham and Marsden, 1987, section 2.1). A plot of sample rays (3.2) is given in Figure 1 (red lines).

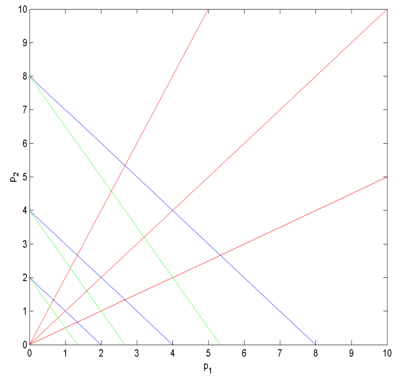

Figure 1. Sample integral curves of Ξ (red rays) and isoexpenditure constraints.

Turn to the action of the group of homotheties on $\mathcal{B}$. Scale transformation on consumption set can be generated by the vector field (compare Mantovi, 2013; Tyson, 2013)

$$\mathbf{Z} = \sum_{k=1}^{n} q^k \frac{\partial}{\partial q^k} \qquad (3.3)$$

Integral curves of (3.3) read ($j = 1, ... , n$, $-\infty < s < +\infty$), for any initial condition $\mathbf{q}(0)$,

$$q^k(s) = q^k(0)\,e^s \qquad (3.4)$$

and one can write $\mathbf{q}(s) = e^{s\mathbf{Z}}\,\mathbf{q}(0)$. Once computed on a scalar function $\psi : \mathcal{B} \to \mathbb{R}$, the vector field (3.3) enables us to define the elasticity of scale of ψ as the ratio $\mathbf{Z}(\psi)/\psi$. Then, for a d-h function $f(\mathbf{q})$, $\mathbf{Z}(f) = d\,f$ (Euler's formula)[3], and correspondingly for d-h functions on $\mathcal{A}$ with respect to Ξ. The structural

[3] Evidently, for any nonvanishing number b, $b\mathbf{Z}$ generates as well the group of homotheties of $\mathcal{B}$; still, it takes $\mathbf{Z}$ to represent Euler's formula as $\mathbf{Z}(f) = d\,f$.

dual role of the flows (3.2) and (3.4) is established by

Proposition 3.1. *The flow of* Ξ *is adapted to the constraint structure on* $\mathcal{A}$ *according to*

$$e^{-s\Xi}\,\alpha(\mathbf{q}) = \alpha(e^{s}\mathbf{q}) \tag{3.5}$$

for any $\mathbf{q}\in\mathcal{B}$ *and* $s\in\mathbb{R}$. *Correspondingly, for any* $\mathbf{p}\in\mathcal{A}$ *and* $s\in\mathbb{R}$, *the flow of* $\mathbf{Z}$ *is adapted to the constraint structure on* $\mathcal{B}$ *according to*

$$e^{s\mathbf{Z}}\,\beta(\mathbf{p}) = \beta(e^{-s}\mathbf{p}) \tag{3.6}$$

Proof. Check the proposition by means of the explicit formulas (3.2) and (3.4) for the integral curves of the vector fields Ξ and $\mathbf{Z}$.

Ξ and $\mathbf{Z}$ are scale dual in the sense of (3.5) and (3.6): the LHSs of (3.5) and (3.6) represent the action of scale transformations on constraints on $\mathcal{A}$ and $\mathcal{B}$ respectively, whereas the RHSs represent the action of scale transformations on the respective dual space.

Admittedly, Proposition 3.1, which does not represent an original result, fixes the explicit representation of the role of scale transformations as acting on the constraint structure of the theory, and therefore of the mechanism underlying the equivalence between utility maximization and expenditure minimization (whose foundational status is thoroughly discussed by Mas-Colell et al., 1995). Recall, such an equivalence is at the basis of Shephard duality: in a sense, all of duality results stem from the mechanism represented in Proposition 3.1. Figure 1 pictures a number of sample unit-expenditure constraints and integral curves of Ξ as representation of the process of dragging along simplexes on $\mathcal{A}$ for $n = 2$.

Given the foundational role of Proposition 3.1, still, the simplicity of its proof may lead one not to focus properly the geometrical mechanism underlying such proposition. In such respects, notice that not every radial flow is adapted to the dual constraint structures. For instance, one can check that, in the case of a pair of consumption goods, the flow of the radial vector field

$$\frac{1}{y}\frac{\partial}{\partial x} + \frac{1}{x}\frac{\partial}{\partial y} \tag{3.7}$$

on the consumption set $(0,\infty)\times(0,\infty)$ does not map simplexes onto simplexes. In fact, being

$$\begin{aligned} x(s) &= x(0)\sqrt{1+\frac{2}{x(0)y(0)}s} \\ y(s) &= y(0)\sqrt{1+\frac{2}{x(0)y(0)}s} \end{aligned} \tag{3.8}$$

the explicit representation of the integral curves of (3.7) for any initial condition $x(0)$, $y(0)$ and $s \in \left(\frac{-1}{2x(0)y(0)}, \infty\right)$, it is quite evident that the linearity of the equations of budget constraints is not preserved by such drag along, being the points of a simplex dragged towards a curved locus.[4]

Remark 3.2. Proposition 3.1 exploits the dual relevance of the condition $\mathbf{pq} = 1$. In fact, as a consequence of Property 2.2, such a condition can be given both base and tangent interpretation. On the one hand, being $\mathbf{q}$ and $\mathbf{p}$ elements of $\mathcal{B}$ and $\mathcal{A}$ respectively, the condition defines the constraint structure subject of Proposition 3.1. On the other hand, the condition $\mathbf{pq} = 1$ can be given dual tangent significance, namely, it can represent the application of $\mathbf{Z}$ to a 1-form $\mathbf{p}(\mathbf{q})$ on $\mathcal{B}$,[5] or it can represent the application of Ξ to a 1-form $\mathbf{q}(\mathbf{p})$ on $\mathcal{A}$. In such respects, we shall be in a position to deepen the nature of the fundamental identities of duality theory (section 6).

We thereby conclude the assessment of the absolute levels of duality. The stage is set for the introduction of the key notion of our approach.

4. Expansion vector field

As pointed out in the introduction, the intuition underlying our approach posits that, under suitable conditions (in first instance, absence of nonconvexities), expansion paths on primal space can be represented as the flow of a vector field, whose parametrization can be employed to gauge INE. Such a global representation of *pure* INE, in principle, enables one to represent expansion paths as solutions to ODE systems (dynamical systems).

A vector field on a manifold $\mathcal{M}$ is a section of the tangent bundle $T\mathcal{M}$, i.e. a function which identifies

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[4] The same exercise can be performed with respect to the drag along of the arcs of circumference $\sum_{k=1}^{n} (q^k)^2 = c$ which foliate $\mathcal{B}$ smoothly. The flow of the vector field (3.3) is adapted to such a foliation (maps arcs onto arcs) whereas the flow of (3.7) is not.

[5] It is not difficult to convince oneself that $\mathcal{B} \times \mathcal{A}$ is diffeomorphic to a proper open subset of the cotangent bundle $T^*\mathcal{B}$: identify $\mathbf{p}\in\mathcal{A}$ with the corresponding element of $T^*_{\mathbf{q}}\mathcal{B}$ according to Property 2.2.

a unique tangent vector at each point of $\mathcal{M}$ (Figure 2). Much like the velocity field in the stationary flow of a fluid represents the velocities of tiny particles dragged by the flow, so the expansion vector field on consumption set represents the rate at which Marshallian demand changes as consumers choices are dragged by increasing optimal expenditure at fixed prices.

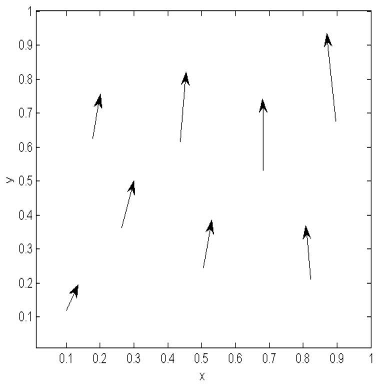

Figure 2. A standard qualitative representation of a vector field in terms of sample vectors. The length of vectors represents the intensity of the field at that point.

The vector field Ξ generates such effects, so that

Proposition 4.1. *For any consumer satisfying Assumption* 2.1 *the class of expansion paths can be represented as the flow of the vector field* $\mathbf{X} \equiv -\varphi_* \Xi = \varphi_*(-\Xi)$, *such that* $\mathbf{X}$(MRS) = 0.

Proof. Marshallian demand φ maps each point of a ray on $\mathcal{A}$ onto a point of the corresponding expansion path on $\mathcal{B}$, so that the parametrization (3.2) of the ray induces a parametrization on the corresponding expansion path. Then, the push-forward φ_* maps each vector tangent to the ray onto a vector tangent to the expansion path, which represents an integral curve of the expansion vector field $\varphi_*(-\Xi)$ (the minus sign guarantees that the vector field is indeed oriented towards expansion), along which MRS are constant. By (3.2) and (3.4), the coordinate components of $\mathbf{X}$ are given by $\frac{dq^k}{dI} I = \frac{dq^k}{d\ln I}$.

Proposition 4.1 fixes a global approach to INE, parametrized according to (3.4). Despite the 'removal' of income from the set of dual variables, we have gained a consistent geometric framework for addressing INE. The income variable, in fact, has not been removed, it has been employed in a change of variables (the projection $\pi : \Delta \to \mathcal{A}$) adapted to the 0-h symmetry of Marshallian demand and to the structure of scale duality. Compare the standard parametrization of INE by means of income employed in Slutsky decompositions, which identify infinitesimal entangled expansion and substitution effects which do commute by definition (see also Mantovi, 2013).

The expansion vector field is a dimensional object (quantity). Being the economic relevance of dimensionless (and then unit-free) measures well established, elasticity concepts are often considered; then, let us connect $\mathbf{X}$ with income elasticity of demand. Evidently, the n functions ln q^k, with differentials (gradients) $\frac{dq^k}{q^k}$, are 'probes' of the scaling structure of $\mathcal{B}$, in that $\mathbf{Z}(\ln q^k) = 1$. Then one can state

Proposition 4.2. *For* $k = 1,\ldots, n$, *income elasticity of demand of good k is given by*

$$\frac{dq^k}{q^k}(\mathbf{X}) \equiv \mathbf{X}(\ln q^k) = X^k(\mathbf{q})\frac{\partial}{\partial q^k}(\ln q^k) = \frac{X^k(\mathbf{q})}{q^k} \quad (4.1)$$

Proof. Given the definition $\frac{dq^k}{dI}\frac{I}{q^k}$ of income elasticity of demand of good k, and the fact that the components of the vector field $\mathbf{X}$ do coincide with $\frac{dq^k}{dI} I = \frac{dq^k}{d\ln I}$ (Proposition 4.1), the pairing (4.1) yields the desired result.

Recall that income elasticity of demand exceeding 1 is the condition for a good of being a luxury good, which can be stated as $X^k(\mathbf{q}) > q^k$. Evidently, not all goods can be luxury goods. The constraint $1 = \sum_{k=1}^{n} P_k \frac{dq^k}{dI}$ is well known to hold for the marginal propensities to consume $\frac{dq^k}{dI}$; correspondingly, the components of $\mathbf{X}$ do satisfy the equivalent identity $1 = \sum_{k=1}^{n} p_k X^k$. Finally, we are in a position to fix a well known benchmark property of homothetic models.

Proposition 4.3. *Given homothetic preferences satisfying Assumption* 2.1, $\mathbf{X} = \mathbf{Z}$. *For any homothetic consumer the curves* (3.4) *represent expansion paths parametrized by s, the natural logarithm of the scale* e^s; *the vector field* $\mathbf{Z}$ *generates the expansion flow of any homothetic consumer with respect to normalized prices.*

Proof. It is well established that Marshallian demand φ is 1-h in income for homothetic models.

Then, with respect to normalized prices, $\varphi\left(\frac{\mathbf{p}}{\lambda}\right)=\lambda\,\varphi(\mathbf{p})$ for any positive λ. Take $\lambda = e^s$ and obtain the integral curves (3.4), which uniquely identify the class of (nonvanishing) vector fields proportional to **Z**. Then, an INE is uniquely determined by the scale factor $\lambda = e^s$.

In fact, there is more to Proposition 4.3 than the well known fact the expansion paths are rays for homothetic consumers. Proposition 4.3 represents expansion paths as parametrized according to the flow (3.4) of **Z**, as an explicit breaking of 0-h of Marshallian demand for all consumers: all homothetic consumers behave the same as far as INE are at stake, and all display unit income elasticity of demand, a result which we can represent by plugging **X** = **Z** into (4.1), so as to obtain

$$\frac{dq^k}{q^k}(\mathbf{Z})=q^k\,\frac{\partial}{\partial q^k}(\ln q^k)=1 \tag{4.2}$$

Thus, as far as INE are under concern, **Z** can be considered a complete characterization of homotheticity, and deviations of the expansion flow from the flow of **Z** can be taken as a measure of deviation from homotheticity. Let us employ a pair of examples in order to enlighten the nature of the expansion vector field.

Example 4.4. Start with the homothetic Cobb-Douglas models $U(\mathbf{q})=\sum_{k=1}^{n} a_k \ln q^k$ with $1=\sum_{k=1}^{n} a_k$, for which Marshallian demand reads $q^k=\frac{a_k}{p_k}$, being p_k normalized prices. Consider INE as parametrized by (3.4). Then, the components of **X** are given by

$$X^k=\frac{d}{ds}q^k(s)\,|_{s=0}=\frac{d}{ds}\frac{a_k}{p_k}e^s\,|_{s=0}=\frac{a_k}{p_k} \tag{4.3}$$

Then, employing the well known expression of inverse demand, one obtains $X^k(\mathbf{q})=q^k$, so that the expansion vector field **X** does indeed coincide with **Z** (compare the expression for the expansion vector field with respect to the choice of a numeraire set forth by Mantovi, 2013).

Example 4.5. Consider the model of quasi-homothetic symmetric preferences discussed by Bertoletti (2006), which do satisfy Assumption 2.1. For the sake of simplicity, consider the case of two goods, and then the direct utility functions

$$U(x,y)=-\tfrac{1}{\alpha}(e^{-\alpha x}+e^{-\alpha y})\ ,\qquad \alpha>0 \tag{4.4}$$

on the consumption set $(0,\infty)\times(0,\infty)$. Optimal consumption is uniquely determined by the FOC

$$\begin{aligned}-\alpha x&=\ln \mathbf{Z}(U)+\ln p_x\\ -\alpha y&=\ln \mathbf{Z}(U)+\ln p_y\end{aligned} \tag{4.5}$$

being $\mathbf{Z}(U)=xe^{-\alpha x}+ye^{-\alpha y}$, from which we derive the cartesian equations

$$x-y=\tfrac{1}{\alpha}\ln\frac{p_y}{p_x} \tag{4.6}$$

for expansion paths, which turn out to be straight lines with unit angular coefficient. In Figure 3 sample indifference curves and expansion paths for α = 1 are represented. Notice, MRS are bounded from above, and indifference curves hit the boundary of consumption set.

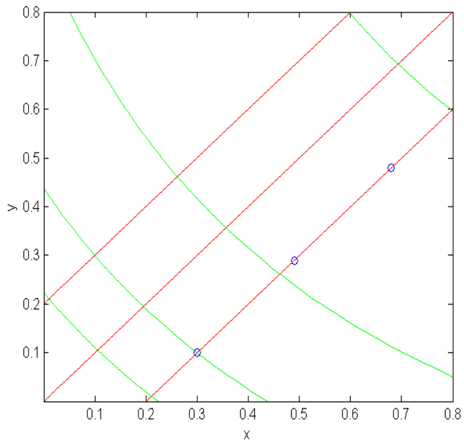

Figure 3. Sample indifference curves (green curves) and expansion paths (red lines) for the model (4.4) for α = 1. The lower dot represents the bundle (0.3,01); the INE corresponding to doubled (tripled) income leads to the bundle represented by the intermediate (upper) dot.

Thus, we expect the expansion vector field to be of the form

$$\mathbf{X}(x,y)=\xi(x,y)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right) \tag{4.7}$$

with $\xi(x,y)=\xi(y,x)$ smooth, in order to generate the expansion paths (4.6). Definitely, employing the Jacobian of Marshallian demand

$$x(p_x,p_y)=\frac{1-\frac{1}{\alpha}p_y\ln\frac{p_x}{p_y}}{p_x+p_y}$$
$$y(p_x,p_y)=\frac{1-\frac{1}{\alpha}p_x\ln\frac{p_y}{p_x}}{p_x+p_y} \qquad (4.8)$$

one can apply Proposition 4.1 and compute straightforwardly the components of **X**; one thereby finds that the expansion vector field reads

$$\mathbf{X}(x,y)=\frac{xe^{-\alpha x}+ye^{-\alpha y}}{e^{-\alpha x}+e^{-\alpha y}}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)$$
$$=\frac{Z(U)}{-\alpha U}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right) \qquad (4.9)$$

and thus turns out indeed to be symmetric in x and y, as well as parallel to the principal diagonal $y = x$, and such that **X**(MRS) = 0 and **X**(U) = **Z**(U). Integral curves of (4.9) read

$$x(s)=x(0)e^s+(1-e^s)\frac{1}{\alpha}\frac{p_y}{p_x+p_y}\ln\frac{p_y}{p_x}$$
$$y(s)=x(s)-\frac{1}{\alpha}\ln\frac{p_y}{p_x} \qquad (4.10)$$

consistently with (4.6). We thereby obtain an explicit global parametrization of pure INE as represented in Figure 3: it is not difficult to employ Marshallian demand (4.8) and its inverse

$$p_x=\frac{e^{-\alpha x}}{xe^{-\alpha x}+ye^{-\alpha y}}\ ,\quad p_y=\frac{e^{-\alpha y}}{xe^{-\alpha x}+ye^{-\alpha y}}\ , \qquad (4.11)$$

to convince oneself that for α = 1 the bundle J = (0.3,0.1)= $\varphi\left(\frac{1}{0.3+0.1e^{0.2}},\frac{1}{0.1+0.3e^{-0.2}}\right)$ is mapped onto $e^{\ln 2\,\mathbf{X}}J \cong$ (0.49,0.29) once income is doubled (INE parametrized by ln2 as value of the flow parameters of Ξ and **X**), and onto roughly (0.68,0.48) once income is tripled. Proposition 4.2 enables one to establish that income elasticity of demand for good x is $\frac{xe^{-\alpha x}+ye^{-\alpha y}}{x(e^{-\alpha x}+e^{-\alpha y})}$ (and correspondingly for y), and then unitary on the principal diagonal. We refer to Pollak (1971) and Bertoletti (2006) for the conceptual relevance of the model.

5. Symmetries of preferences

Tyson (2013) sets forth a thorough analysis of discrete and continuous symmetries of preferences, and employs vector fields on primal space as generators of 1-parameter groups of symmetry transformations preserving the ordering defined by the preferences under consideration. Such symmetry flows map indifference sets onto indifference sets, and then MRS are constant along such flows. Being MRS constant along expansion paths, one envisions a fundamental connection between symmetry vector fields and expansion vector fields. Let us fix such a connection.

Proposition 5.1. *Given Assumption* 2.1*, a symmetry vector field* (*in the sense of Tyson,* 2013) *is parallel to the expansion vector field, so that a 'conversion' factor exists between them.*

Proof. A symmetry vector field is such that MRS are constant along its flow, so that it is everywhere parallel to the expansion vector field. Then, at any point, they are proportional one to another: the point dependent conversion factor defines a smooth function on consumption set. In fact, given a symmetry vector field, any affine reparametrization thereof is again a symmetry vector field, and the relative conversion functions are obtained one from another by constant scaling.

The well behavior of preferences in the above argument is crucial: continuous symmetries exist for nonconvex problems (see Tyson, 2013), which cannot be connected simply with expansion paths as in the previous proposition, in that expansion paths do not 'span' the whole of consumption set.

Tyson (2013) addresses the properties of PDEs associated with continuous symmetries of preferences: if **Y** is a symmetry vector field for a definite class of equivalent utility functions, then, for $1 \le i < j \le n$, the conditions **Y**(MRS_{ij}) = 0 define PDEs for a representative utility function. On account of Proposition 5.1, such equations can be written **X**(MRS_{ij}) = 0.

In fact, Proposition 5.1 points at a fundamental connection in the theory of differential equations. Notice, the expansion vector field defines both the ODE system for expansion paths as well as PDEs for functions having a define property along such paths (for instance, the property of being constant). A similar link connects for instance a hamiltonian vector field with the PDE characterizing constants of the motion, namely, the vanishing of the Poisson bracket between the constant of the motion and the hamiltonian function (Abraham and Marsden, 1987). Thus, Proposition 5.1 establishes a deep geometrical level for the link between INE and symmetries. True, from the economic standpoint, the analysis of a few transparent examples pays a lot.

Example 5.2. Consider the preferences represented by the direct utility function

$$U(x, y) = y + \ln x \tag{5.1}$$

on the consumption set $(0,\infty)\times(0,\infty)$, which do satisfy Assumption 2.1. The functional form (5.1) displays a continuous symmetry, namely, translation along *y*: as represented in Figure 4, indifference curves can be obtained one from another by vertical translation (see also Silberberg, 1972). Such a simple symmetry sets an ideal setting for discussing the link with expansion effects.

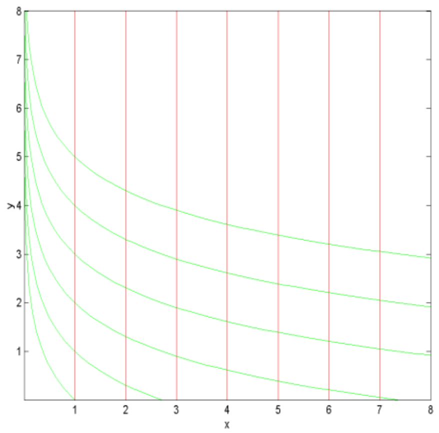

Figure 4. Sample indifference curves (green curves) and expansion paths (red lines) for the model (5.1).

Optimal consumption is uniquely determined by the FOC

$$\begin{aligned} \frac{1}{x\mathbf{Z}(U)} &= p_x \\ \frac{1}{\mathbf{Z}(U)} &= p_y \end{aligned} \tag{5.2}$$

being $\mathbf{Z}(U) = 1 + y$, and we derive simply the cartesian equations

$$x = \frac{p_y}{p_x} \tag{5.3}$$

for expansion paths, which turn out to be straight lines parallel to the *y* axis, along which MRS are constant. Thus we expect the expansion vector field to be of the form $f(x, y)\frac{\partial}{\partial y}$. In fact, the vector field $\frac{\partial}{\partial y}$ is a symmetry vector field for the preferences represented by (5.1): being

$$\begin{aligned} x(s) &= x(0) \\ y(s) &= y(0) + s \end{aligned} \tag{5.4}$$

the integral curves of $\frac{\partial}{\partial y}$ for any initial condition and $s \in (-y(0), \infty)$, it is not difficult to convince oneself that indifference curves are mapped onto indifference curves by the flow of $\frac{\partial}{\partial y}$.

Along the lines employed in example 4.5, one can compute the expansion vector field by employing Marshallian demand

$$x(p_x, p_y) = \frac{p_y}{p_x}\ ,\quad y(p_x, p_y) = \frac{1}{p_y} - 1 \tag{5.5}$$

and its inverse

$$p_x = \frac{1}{x(1+y)}\ ,\quad p_y = \frac{1}{1+y} \tag{5.6}$$

and find

$$\mathbf{X}(x, y) = (1+y)\frac{\partial}{\partial y} \tag{5.7}$$

with integral curves

$$\begin{aligned} x(s) &= x(0) \\ y(s) &= y(0)e^s - 1 \end{aligned} \tag{5.8}$$

for $s \in (-\ln y(0), \infty)$. Thus, the flow of $\mathbf{X}$ does *not* map indifference curves onto indifference curves. Still, one can 'reparametrize' $\mathbf{X}$ and define the vector $\frac{1}{1+y}\mathbf{X}$, which coincides with symmetry vector field $\frac{\partial}{\partial y}$. Notice that $\mathbf{X}(U) = \mathbf{Z}(U)$, in perfect analogy with example 4.5.

Example 5.3. Consider the preferences represented by (4.4). It is not difficult to convince oneself that for any non vanishing real number λ the vector field $\lambda\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$ is a symmetry vector field for such preferences, in that its flow maps indifference curves onto indifference curves. On account of (4.9), the conversion factor yielding the expansion vector field is easily determined.

The previous examples enlighten the link between INE and symmetries: INE do not map indifference curves onto indifference curves, since they reflect the *absolute* scaling structure of the theory (Proposition 3.1). On the other hand, symmetries are not a structural property of the theory; still, once a class of equivalent utility functions admits a continuous symmetry (in the sense of Tyson), a link with INE exists as discussed above. This is a major justification (at least in the author's view) for deepening the differential aspects of duality.

Definitely, Tyson (2013) addresses the homothetic symmetry among the many others, and does not posit a preferred role for such symmetry. With a different attitude, we shall deepen the link between expansion and symmetries in section 7.

6. Demand and conjugation

Expansion paths do not embody all of the traits of a consumption problem; for instance, all homothetic consumers behave the same as long as INE are at stake. Still, the expansion vector field turns out to represent a sound probe of the properties of a model, as emerges from the following discussion of basic model-dependent elements of duality theory, namely, Hotelling-Wold (HW) and Antonelli-Roy (AR) identities and conjugation.

6.1 Fundamental identities

Our geometric setup enables us to frame consistently the HW and AR identities. Recall, with respect to normalized prices, for j = 1,..., n, the HW identities

$$p_j(\mathbf{q}) \sum_{k=1}^{n} q^k \frac{\partial U}{\partial q^k}(\mathbf{q}) = \frac{\partial U}{\partial q^j}(\mathbf{q}) \qquad (6.1)$$

provide an explicit representation of inverse Marshallian demand, whereas, for k = 1,..., n, the AR identities

$$q^k(\mathbf{p}) \sum_{j=1}^{n} p_j \frac{\partial V}{\partial p_j}(\mathbf{p}) = \frac{\partial V}{\partial p_k}(\mathbf{p}) \qquad (6.2)$$

provide an explicit representation of Marshallian demand. By identifying the q^k and p_j with the corresponding components of vectors tangent to $\mathcal{B}$ and $\mathcal{A}$ respectively, one is in a position to state

Proposition 6.1. *Given Assumption* 2.1, *the identities* (6.1) *represent the components of the identity*

$$dU(\mathbf{q}) = \mathbf{Z}(U(\mathbf{q}))\, \varphi^{-1}(\mathbf{q}) \qquad (6.3)$$

between 1-forms on $\mathcal{B}$*. Correspondingly, the identities* (6.2) *represent the components of the identity*

$$dV(\mathbf{p}) = \Xi(V(\mathbf{p}))\, \varphi(\mathbf{p}) \qquad (6.4)$$

between 1-forms on $\mathcal{A}$ *, so that* (6.3) *is the push-forward* φ_* *of* (6.4).

Proof. The standard proof (Cornes, 1992) of (6.1) and (6.2) fixes the independence of such identities from the utility representation. Then, LHSs and RHSs of such identities can be consistently interpreted as components of 1-forms. For instance, dU is an exact 1-form on $\mathcal{B}$, $\mathbf{p}$ is a 1-form on $\mathcal{B}$ (Property 2.2), and $\mathbf{Z}(U)$ represents the 'conversion factor' (function) between the two. Analogous arguments hold for AR identities. By definition, the LHS of (6.3) is the push-forward φ_* of the LHS of (6.4) provided $V = \varphi^*U$.

The relevance of HW and AR identities is typically attributed to their being explicit representations of direct and inverse Marshallian demand. In fact, the geometric status fixed by Proposition 6.1 provides a consistent framework for employing such identities between 1-forms on $\mathcal{B}$ and $\mathcal{A}$ respectively, and fixes a basic property of $\mathbf{Z}(U)$: being dU and $\mathbf{p}$ both 1-forms on $\mathcal{B}$ (Property 2.2), $\mathbf{Z}(U)$ is the 'conversion factor' between the two. Notice that both dU and $\mathbf{Z}(U)$ depend of the utility representation, whereas their ratio does *not*. Furthermore, being U 1-h, so that $\mathbf{Z}(U)$ = 1, one has $dU = \mathbf{p}$, which represents the scale symmetry of the FOC representing optimal homothetic consumption. Perfectly symmetric arguments hold for dV and $\mathbf{q}$.

As a glance into the effectiveness of our geometric description, consider the following ansatz about the form of the geometric HW identity. Forget about (6.1) and ask what the conversion factor $\chi(\mathbf{q})$ between the 1-forms dU and $\mathbf{p}$ should be. Definitely, in the light of our remarks on tangent duality, the equilibrium condition $\mathbf{pq}$ = 1 implies that applying $\mathbf{Z}$ to both sides of $dU(\mathbf{q}) = \chi(\mathbf{q})\mathbf{p}$ we obtain $\mathbf{Z}(U) = \chi$. A corresponding dual argument holds.

The expansion vector field can be applied to both sides of (6.3) and yield (disregarding the dependence on $\mathbf{q}$ for the sake of simplicity of the formula)

$$dU(\mathbf{X}) = \mathbf{X}(U) = \mathbf{Z}(U)\, \varphi^{-1}(\mathbf{X}) \qquad (6.5)$$

Thus, the pairing between $\mathbf{X}(\mathbf{q})$ and $\varphi^{-1}(\mathbf{q})$ equals the ratio between $\mathbf{X}(U)$ and $\mathbf{Z}(U)$. Consistency of (6.5) with the constraint $1 = \sum_{k=1}^{n} p_k X^k$ for the marginal propensities to consume implies

Proposition 6.2. *Given Assumption* 2.1, *for any direct utility function U, we have* $\mathbf{X}(U) = \mathbf{Z}(U)$, *so that* $\varphi^{-1}(\mathbf{X}) = 1$. *Then, the vector field* $\mathbf{X}$–$\mathbf{Z}$ *is tangent to indifference sets.*

Recall Remark 3.2 and notice that Proposition 6.2 enables us to appreciate the tangent significance of expressions of the form $\mathbf{pq}$. Despite the different (in general) directions of $\mathbf{X}$ and $\mathbf{Z}$, their application to the utility function yields the same result, as a consequence of the properties of optimization embodied by **X**, and of the parametrization of **X** induced by Ξ. Notice that for homothetic models (6.5) reduces to $\mathbf{Z}(U) = \mathbf{Z}(U)\ \varphi^{-1}(\mathbf{Z})$, i.e. $1=\varphi^{-1}(\mathbf{Z})$. For instance, employing the inverse demand $p_k = \frac{a_k}{q^k}$ of Cobb-Douglas models we obtain $\varphi^{-1}(\mathbf{X}) \equiv \sum_{k=1}^{n} \varphi_k^{-1}(\mathbf{q}) X^k(\mathbf{q}) = \sum_{k=1}^{n} \frac{a_k}{q^k} q^k = 1$. In addition, notice that we have checked explicitly that the expansion vector fields (4.9) and (5.7) are such that $\mathbf{X}(U) = \mathbf{Z}(U)$.

6.2 Conjugation and expansion

The Shephard duality between distance and expenditure functions is typically expounded as the symmetry linking the optimization problems which define such functions: much like $\mathcal{D}(\mathbf{q},u)$ coincides with the minimum of **pq** for unit expenditure, so $\mathcal{E}(\mathbf{p},u)$ coincides with the minimum of **pq** for unit distance (for instance, Cornes, 1992, pp. 76-77). Let us envision $\mathcal{D}$ and $\mathcal{E}$ as parametrization of the scaling flows, and thereby pave the way for a transparent use the expansion vector field of in the characterization of departure from conjugation as homothetic benchmark.

On account of Assumption 2.1, the indifference sets on $\mathcal{A}$ ($\mathcal{B}$) define a foliation of $\mathcal{A}$ ($\mathcal{B}$). We are thereby guaranteed that any element of $\mathcal{A}$ ($\mathcal{B}$) belongs to an element of the indifference foliation of $\mathcal{A}$ ($\mathcal{B}$). Then, the vector field **Z** can be employed for the definition of $\mathcal{D}(\mathbf{q},u)$: the distance between the bundle **q** and the utility level u can be defined in terms of the parameter interval which connects **q** with the point along the curve (3.4) belonging to the indifference set $U(\mathbf{q}) = u$, i.e.

$$U\left(e^{-\ln \mathcal{D}(\mathbf{q},u)\mathbf{Z}}\,\mathbf{q}\right) = U\left(\frac{\mathbf{q}}{\mathcal{D}(\mathbf{q},u)}\right) \equiv u \qquad (6.6)$$

(see for instance Gorman, 1976; Deaton, 1979) irrespective of the utility representation. Let us write $e^{-\ln \mathcal{D}(\mathbf{q},u)\mathbf{Z}}\mathbf{q}$ a shorthand for (3.4) with $s(\mathbf{q},u)=-\ln\mathcal{D}(\mathbf{q},u)$. 1-h of $\mathcal{D}$ in **q** can be written $\mathbf{Z}(\mathcal{D}(\mathbf{q},u)) = \mathcal{D}(\mathbf{q},u)$. Corresponding to (6.6), being $V(\mathbf{p})$ indirect utility as function of normalized prices (the pull-back of U via Marshallian demand), the expenditure function $\mathcal{E}(\mathbf{p},u)$[6] is such that

$$V\left(e^{-\ln \mathcal{E}(\mathbf{p},u)\Xi}\,\mathbf{p}\right) = V\left(\frac{\mathbf{p}}{\mathcal{E}(\mathbf{p},u)}\right) \equiv u \qquad (6.7)$$

with $\Xi(\mathcal{E}(\mathbf{p},u)) = \mathcal{E}(\mathbf{p},u)$. Thus, we succeed in defining distance and expenditure functions as global logarithmic parametrizations of the flows of **Z** and Ξ, given the indifference sets $\mathcal{A}|_u$ and $\mathcal{B}|_u$ as the loci of initial conditions transverse to the flows (i.e. each integral curve intersects the locus at a single point). For any u belonging to the range $K = U(\mathcal{B})$ of U (which, evidently, coincides with the range of its pull-back V), $\mathcal{E}$ is a function on $\mathcal{A}$ and $\mathcal{D}$ is a function on $\mathcal{B}$. We are in a position to introduce the conjugation property defined by Gorman (1976).

Recall, the bundle **q** and the price vector **P** are conjugate at the utility level u if $\mathcal{D}(\mathbf{q},u)\mathcal{E}(\mathbf{P},u) = \mathbf{Pq}$. Then, being normalized prices the dual variables, $\mathbf{q}\in\mathcal{B}$ and $\mathbf{p}\in\mathcal{A}$ are u-conjugate if $\mathcal{D}(\mathbf{q},u)\mathcal{E}(\mathbf{p},u) = \mathbf{pq}$. Evidently, if $\mathcal{D}(\mathbf{q},u) = 1$, then **q** and $\varphi^{-1}(\mathbf{q})$ are u-conjugate. In general, the inequality (Gorman, 1976; Deaton, 1979; Cornes, 1992)

$$\mathcal{D}(\mathbf{q},u)\mathcal{E}(\mathbf{p},u) \le \mathbf{pq} \qquad (6.8)$$

embodies the convexity of the economic problem. Such an inequality is well known to reduce to equality for homothetic problem, in the sense of

Property 6.3. *Given homothetic preferences satisfying Assumption* 2.1, *if* $\mathbf{q}\in\mathcal{B}$ *and* $\mathbf{p}\in\mathcal{A}$ *are conjugate at* $u\in K$, *they are conjugate at any* $u\in K$.

Recall that it is trivial to check that **p** and $\varphi(\mathbf{p})$ are conjugate at any u for homothetic models, and that in order to prove the above property one can rely on the factorization of distance ad expenditure functions. In fact, the conjugation relation is a binary symmetric relation between rays on dual spaces, and it is no wonder that such a relation displays the benchmark homothetic limit fixed by Property 6.3.

Example 6.4. Consider $a\in(0,1)$ and the Cobb-Douglas utility function $U(x,y) = x^a y^{(1-a)}$, whose associated expenditure function is well known to read $\mathcal{E}(p_x,p_y;u) = u\left(\frac{p_x}{a}\right)^a \left(\frac{p_y}{1-a}\right)^{1-a}$. It is a standard exercise to employ the inverse Marshallian demand $p_k = \frac{a_k}{q^k}$ in order to write $\mathcal{D}(x,y;u) = \frac{1}{\mathcal{E}(p_x(x,y),p_y(x,y);u)} = \frac{x^a y^{1-a}}{u}$. We thereby check that for the homothetic Cobb-Douglas models the identity $\mathcal{D}(\mathbf{q},u)\mathcal{E}(\varphi^{-1}(\mathbf{q}),u) = 1$ holds true for any $u\in(0,\infty)$.

Then, consider the case $a = 0.5$, $\mathbf{p} = (1,0.25)\in\mathcal{A}$ and $\mathbf{q} = (1,4)\in\mathcal{B}$, which differs from φ (1,0.25) =

[6] Being the expenditure function 1-h in prices, the identity $\mathcal{E}(\mathbf{P},u) = I\ \mathcal{E}(\mathbf{p},u)$ (Cornes, 1992, p. 94) connects the representation $\mathcal{E}(\mathbf{P},u)$ in terms of prices with the representation $\mathcal{E}(\mathbf{p},u)$ in terms of normalized prices .

(0.5,2), and is such that **pq** = 2. Being $\mathcal{D}(\mathbf{q},1)$ = 2 and $\mathcal{E}(\mathbf{p},1)$ =1, **q** and **p** are conjugate at u=1. Employ the expression of Cobb-Douglas distance and expenditure functions in order to check that **q** and **p** are conjugate at any $u \in (0,\infty)$.

The departure from conjugation defines quite a natural setting for envisioning the nature of the expansion vector field **X**: rays in the space of normalized prices can be taken as parametrization of INE, so that it is natural to expect **X** to provide a differential representation of the departure from Property 6.3.

Let us specialize inequality (6.8) to equilibrium, i.e.

$$\mathcal{D}(\varphi(\mathbf{p}),u)\mathcal{E}(\mathbf{p},u) \le \mathbf{p}\varphi(\mathbf{p}) \tag{6.9}$$

Standard arguments supporting (6.8), and then (6.9), exploit the convexity of $\mathcal{D}$ and $\mathcal{E}$. We are in a position to employ the properties of **X** in order to envision the inequality

$$\mathbf{X}(\mathcal{D}(\mathbf{q},u)) \le \mathcal{D}(\mathbf{q},u) = \mathbf{Z}(\mathcal{D}(\mathbf{q},u)) \tag{6.10}$$

as a differential counterpart to (6.9).

As represented in Figure 5, one can take $\mathbf{q}_0$ as initial condition for both the flows of **Z** and **X**. Consider the same value s = ln2 for the parameter flows, and then $2\mathbf{q}_0 = e^{\ln 2\mathbf{Z}}\,\mathbf{q}_0$ and $\mathbf{q} = e^{\ln 2\mathbf{X}}\,\mathbf{q}_0$.

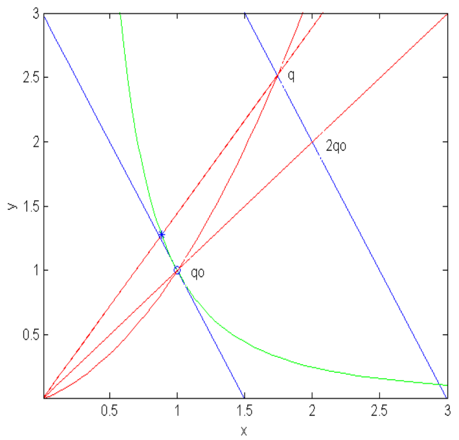

Figure 5. A qualitative representation of the mechanism underlying (6.10). The bundle $2\mathbf{q}_0$ is given by the flow of **Z** with initial condition $\mathbf{q}_0$ and parameter value ln2. The bundle **q** is given by the flow of **X** with initial condition $\mathbf{q}_0$ and parameter value ln2. The curvature of the indifference curve through $\mathbf{q}_0$ is the source of the inequality $\mathcal{D}(\mathbf{q},u) \le \mathcal{D}(2\mathbf{q}_0,u) = 2$.

Figure 5 displays the reason for strict (quasi-) convexity of preferences to imply

$$\mathcal{D}(\mathbf{q},u) \le \mathcal{D}(2\mathbf{q}_0,u) = 2 \tag{6.11}$$

whose differential correspondent is (6.10), since the inequality in (6.11) holds for any value s of the parameter flows.

We thereby envision the departure of the flows of **X** and **Z** as both a measure of departure from homotheticity as well as a measure of departure from conjugation. Then, $\mathcal{D}(\mathbf{q},u)$ can be considered as an 'integral' of $\mathbf{X}(\mathcal{D}(\mathbf{q},u))$, and one recovers the LHS of (6.9) as the 'infinite sum' of 'infinitesimal' contributions of the form $ds\,\mathbf{X}(\mathcal{D}(\mathbf{q}(s),u))\mathcal{E}(\mathbf{p}(s),u)$ at fixed u.

7. Homothetic models

Proposition 4.3 and Property 6.3 above represent well known aspects of homothetic models. True, the preferred status of homothetic models has long been recognized. For instance, Gorman (1976) points out "how very much easier it is to move back and forward between primal and dual representatives if the original utility function is homothetic". In fact, according to Chambers and Mitchell (2001), "Homotheticity may be the most common functional restriction employed in economics." The factorization of distance and expenditure functions is typically taken as the most significant distinguishing feature of homothetic models. The benchmark role of homotheticity in welfare analysis is the subject of Chipman and Moore (1980). More recently, Mantovi (2013) sets forth a characterization of homothetic models in terms of the commutation of expansion and substitution effects represented by means of flows on consumption set. Along similar lines, building on Proposition 4.3, let us employ the commutation properties of scaling and expansion vector fields in order to deepen the benchmark nature of homothetic preferences.

As is well known, the Lie bracket [**A**,**B**] of the vector fields **A**, **B** is a local measure of the commutativity of the associated flows. The vanishing of [**A**,**B**] on an open set O of the manifold implies that within O the flows of **A** and **B** do commute, i.e. starting from any point of O one can follow the flow of **A** for a parameter interval a and then the flow of **B** for a parameter value b and obtain the same final point which is obtained once the flows are followed in the reversed order (see Spivak, 1999). Evidently, coordinate vector fields do commute, since one can follow coordinate lines in any desired order and reach the point uniquely determined by a specified coordinate n-tuple.[7]

[7] More generally, Frobenius theory establishes the integrability of involutive distributions.

To the author's knowledge, Lie brackets do not as yet represent a standard method in microeconomic analysis. Still, Bogetoft et al. (2006) set forth a reversed decomposition of technical and allocative efficiency which can be traced to the mechanism above, once we recognize that technical efficiency can be measured by the flow of the scaling vector field **Z**, and that allocative efficiency can be measured by the flow of a substitution vector field as in Mantovi (2013). Bogetoft et al. (2006), among other things, establish that, for a single output producer, standard and reversed decompositions of overall technical efficiency do coincide if and only if the production function is homothetic. Thus, it is natural to argue about the relevance of a systematic framework for investigating the commutativity of effects for the consumer as well, for which the disparity between willingness to pay and willingness to accept (see for instance Weber, 2010, and references therein) may represent a natural playground.

The following property is meant to fix a geometric approach for such a vision.

Property 7.1. *The Lie bracket (vector field on $\mathcal{B}$)*

$$[\mathbf{Z},\mathbf{X}] \qquad (7.1)$$

provides a consistent measure of departure of preferences from homotheticity in that (7.1) *vanishes if and only if preferences are homothetic. The coordinate representation of* (7.1) *reads*

$$[\mathbf{Z},\mathbf{X}]=\sum_{k=1}^{n}\left(\mathbf{Z}(X^k)-X^k\right)\frac{\partial}{\partial x^k} \qquad (7.2)$$

Proof. On account of Proposition 4.3, (7.1) vanishes if preferences are homothetic: being the Lie bracket skew-symmetric, [**Z**,**Z**] vanishes. If preferences are non homothetic, the vector field (7.1) accounts for the deviation of the flows of **Z** and **X** as a basic principle in the theory of dynamical systems. In fact, it has been long established on economic grounds that expansion paths are not rays for non homothetic models. One can check (7.2) directly by means of the standard formula for the components of the Lie bracket (Spivak, 1999, p. 153); the interpretation of (7.2) is enlightening: the coordinate components of [**Z**,**X**] represent the difference between application of **Z** to the components of **X** and the components of **X** themselves.

Example 7.2. Let us employ the preferences (4.4) in order to enlighten the economic content of the somewhat technical Property 7.1. We expect the computation of (7.1) with **X** given by (4.9) to result in a nonvanishing vector field, being the model not homothetic. In fact, (7.1) results in

$$[\mathbf{Z},\mathbf{X}]=\left((1-\alpha)\xi(x,y)-\alpha\xi^2(x,y)\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right) \qquad (7.3)$$

which is a vector field parallel to expansion vector field (4.9). Then a scale transformation (generated by **Z**) followed a INE (generated by **X**) is not equivalent the reversed sequence of effects, and the 'difference' is parallel to **X**: in this model the *differential* measure (7.1) can be integrated along the flow of **X** and thereby yield a *finite* measure (compare Mantovi, 2013, Appendix 2), corresponding to an INE.

One can check that the effects generated by (7.3) are negligible across the whole consumption set. Non negligible effects are obtained in

Example 7.3. For the expansion vector field (5.7), the Lie bracket (7.1) results in

$$[\mathbf{Z},\mathbf{X}]=\frac{\partial}{\partial y} \qquad (7.4)$$

and then parallel to the expansion vector field as in the previous example. Recall, we have seen that $\frac{\partial}{\partial y}$ is a symmetry vector field for the preferences (5.1), so that in this model the noncommutativity of scale effects and expansion effects is generated by a symmetry vector field.

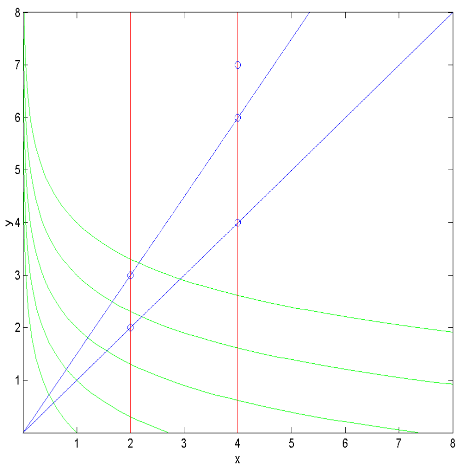

Figure 6. The bundle (2,2) and its images under the scale and expansion effects and their composition in both orders discussed in the main text.

Figure 6 represents an instance of such a noncommutativity. Start with the bundle (2,2). Double the bundle (scale effect) and obtain (4,4), then apply the INE corresponding to doubling income (expansion effect) and obtain (4,7). Then, compose the same effects in reversed order and obtain the final bundle (4.6). Thus, the 'difference' between the bundles (4,7) and (4,6) is a finite correspondent of the differential measure (7.4).

The Lie bracket (7.1) is a vector field on $\mathcal{B}$ which can be computed on any scalar function on $\mathcal{B}$; true, in order to enlighten the significance of (7.1), one has apply it to significant functions, in first instance MRS. Definitely, given $1 \le i < j \le n$, one has

$$[\mathbf{Z},\mathbf{X}](\mathrm{MRS}_{ij}) \equiv \mathbf{Z}(\mathbf{X}(\mathrm{MRS}_{ij})) - \mathbf{X}(\mathbf{Z}(\mathrm{MRS}_{ij})) = -\mathbf{X}(\mathbf{Z}(\mathrm{MRS}_{ij})) \tag{7.5}$$

since MRS are constant along the integral curves of **X**. We thus obtain the derivative along the expansion flow of **Z** applied to MRS. On account of our discussion in section 5, formula (7.5) can be connected with the analysis of symmetries: provided a continuous symmetry of preferences exists, the symmetry vector field can be transformed into the expansion vector field by means of a conversion factor, and then the Lie bracket (7.5) expressed in terms of the symmetry vector field.

As pointed out in section 5, Tyson (2013) does not posit a preferred role for the homothetic symmetry; true, the Lie bracket (7.5) embodies the benchmark role of the homothetic symmetry, which is a symmetry of the structure itself of the theory (in the sense of Proposition 3.1) before than a symmetry of definite preferences.

Then one can employ the arguments in subsection 6.2 in order to argue about the significance of the application of the commutator [**Z**,**X**] to the distance function, such that

$$[\mathbf{Z},\mathbf{X}](\mathcal{D}) = \mathbf{Z}(\mathbf{X}(\mathcal{D})) - \mathbf{X}(\mathcal{D}) \tag{7.6}$$

Notice the 'similarity' between (7.2) and the RHS of (7.6), which holds for any 1-h function.

True, MRS and distance functions embody model dependent features, and therefore the significance of (7.5) and (7.6) can be naturally considered as pertaining to the model-dependent level of the theory. On the other hand, one can compute the vector field (7.1) on functions which do not contain information about a specific problem, and still obtain valuable relations, since the vector field **X** embodies the solution of the consumption problem as far as INE are at stake. For instance, one can consider functions which probe the scaling structure of the theory: true, as already pointed out, the n functions $\ln q^k$ are probes of the scaling structure of primal space, in that $\mathbf{Z}(\ln q^k) = 1$, so that $\mathbf{Y}(\mathbf{Z}(\ln q^k)) = 0$ for any vector field **Y**. Thus, computing (7.1) on such functions we obtain

$$[\mathbf{Z},\mathbf{X}](\ln q^k) \equiv \mathbf{Z}(\mathbf{X}(\ln q^k)) - \mathbf{X}(\mathbf{Z}(\ln q^k)) = \mathbf{Z}(\mathbf{X}(\ln q^k)) \tag{7.7}$$

i.e. we obtain **Z** applied to income elasticity of demand (Proposition 4.2), which, evidently, vanishes for homothetic preferences.

Then, one can apply (7.1) to the 1-form $\varphi^{-1}(\mathbf{q})$ guest of Proposition 6.1, whose coordinate components are the inverse Marshallian demand of the consumption goods. On account of Proposition 6.2, the pairing $\varphi^{-1}([\mathbf{Z},\mathbf{X}])$ vanishes. Then, the line integral of φ^{-1} vanishes once computed on sequences of portions of integral lines of **Z** and **X** with corresponding parametrizations. On account of Propositions 6.1 and 6.2, such a result is equivalent to

$$[\mathbf{Z},\mathbf{X}](U) \equiv \mathbf{Z}(\mathbf{X}(U)) - \mathbf{X}(\mathbf{Z}(U)) = (\mathbf{Z}-\mathbf{X})(\mathbf{Z}(U)) \tag{7. 8}$$

The non vanishing measures (7.5)-(7.8) represent deviation form homotheticity, yet, they do not tell 'how much' deviation. True, once a flexible form for utility functions is employed which reduces to homotheticity in a definite limit, one can define an 'extent' of deviation from homotheticity.

As the reader should expect, we can pull-back Property 7.1 onto its dual correspondent.

Proposition 7.4. *The Lie bracket (vector field on* $\mathcal{A}$*)*

$$[\Xi,\varphi^{*}\mathbf{Z}] \tag{7.9}$$

provides a consistent measure of departure from homotheticity in that (7.9) *vanishes if and only if preferences are homothetic. The coordinate representation of* (7.9) *reads*

$$[\Xi,\varphi^{*}\mathbf{Z}] = \sum_{k,j=1}^{n}\left(\Xi\left(\frac{\partial(\varphi^{-1})_j}{\partial q^k}q^k\right) - \frac{\partial(\varphi^{-1})_j}{\partial q^k}q^k\right)\frac{\partial}{\partial p_j} \tag{7.10}$$

Proof. On account of Proposition 3 in Spivak (1999, p. 190), (7.9) is the pull-back φ^{*} of (7.1). Employ Proposition 4.3 in order to check that (7.9) vanishes if and only if the problem is homothetic. Then, pull-back (7.2) and obtain (7.10).

On account of Proposition 7.4, dual formulas to (7.5)-(7.8) establish a perfectly symmetric dual framework for employing Lie brackets in the characterization of departure from homotheticity.

8. Perspectives

In the last decades the theory of duality has made its way to the foundations of microeconomics, on account of both the fundamental results thereby established, and of the recognition of the pregnancy of 'thinking dual'. For instance, according to Cornes (1992), duality theory conveys "much more direct insight" into problems, and favours "more creative use" of optimization techniques. In such respects, we have been arguing about the effectiveness of a differential geometric approach to duality shaped by the fundamental role of the group of homotheties on both dual spaces. In connection with a natural hypothesis of well behavior, we have been in a position to deepen the differential level of the theory concerning preference symmetries, fundamental identities and conjugation in connection with the benchmark nature of homothetic models.

The limited size of the paper has constrained our analysis to a few elements of duality theory. Still, in the author's vision, the approach can be extended to the various elements of the theory (for instance Slutsky equations), as well as translated to production analysis along well established lines. In fact, duality is often addressed in terms of production problems (for instance, Diewert, 1982; Chambers, 1988).

Notice, we have not been exploiting the freedom of choosing coordinates at will on primal and dual spaces, which is not a key problem in duality theory on account of the economic significance of standard coordinates, namely quantities (of consumption goods or production inputs) and prices. Still, even at fixed coordinates, the geometric characterization of basic principles has long entered the microeconomic inquiry (see for instance, Debreu, 1976; Smale, 1982), in first instance for the purpose of global analysis, which enables one to set free from the straitjacket of comparative statics.[8] In fact, evidently, coordinate changes are required for dealing with characteristics and hedonic prices.

A number of lines of progress for our approach can be envisioned. In section 7 we have been deepening the technical aspects underlying the benchmark nature of homothetic models. A natural line of progress pertains to the transposition of the analysis in section 7 to the problem of productive efficiency. In such respects, notice that Examples 7.2 and 7.3 above can be interpreted in terms of the commutativity of measures of technical efficiency and output expansion. The noncommutativty of efficiency measures discussed by Bogetoft et al. (2006) represents a landmark in such respects. See the discussion on the different interpretations of standard and reversed efficiency measures; among other things, the Authors notice that "it may be advantageous to choose a particular path to obtain overall efficiency if the process has to be carried out sequentially over a period of time" (ivi, p. 460).

Our approach can be employed in the study of separability. For instance, Tyson (2013) employs symmetry vector fields in order to address univariate and multivariate separability. Our geometric framework may extend the dual reach of such an approach in both consumption and production settings. Notice, the dual perspective on separability is well known to provide relevant results (see for instance Blackorby et al., 1978).

The duality properties of benefit functions (Luenberger, 1992) may provide a natural playground for our approach. Recall, benefit functions generalize distance functions, and as such display a duality with expenditure functions, once the reference bundle is property taken into account. Our geometric framework is flexible enough to deal with such an approach, and address for instance translation homotheticity (Chambers and Färe, 1998).

Mantovi (2013) employs the Lie bracket in order to measure the commutativity of expansion and substitution effects, thereby deepening the benchmark nature of homothetic models, and in fact the potentialities inherent to the representation of expansion and substitution effects in terms of flows, i.e. dynamical systems. Our geometric approach to duality is inherently adapted to such a framework, and may contribute to enhance its potentiality.

Finally, in the author's vision, a most promising avenue of progress for our approach is represented by a hamiltonian formulation of duality. As already pointed out, $\mathcal{B} \times \mathcal{A}$ can be considered an open subset of the cotangent bundle $T^*\mathcal{B}$, upon which to setup hamiltonian dynamics. It is quite natural then to conjecture the effectiveness of addressing general models and effects on symplectic grounds, employing Poisson brackets as a realization of Lie brackets we have been discussing, so as to exploit the power of canonical transformations.

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[8] According to Baumol (1973), "We have become used to comparative statics arguments whose results are remarkable for their banality".

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